Generating functional approach to dynamical systems with colored-noise interactions

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References

- S. Suweis, F. Ferraro, C. Grilletta, S. Azaele, A. Maritan
  Generalized Lotka-Volterra Systems with Time Correlated Stochastic Interactions
  Physical Review Letters 133 (16), 167101

- F. Ferraro*, C. Grilletta*, A. Maritan, S. Suweis, S. Azaele
  Exact solution of Dynamical Mean-Field Theory for a linear system with annealed disorder
  arXiv:2405.05183

Background: Lotka-Volterra equations

- Non-linear model for dynamics of ecological community

$$\dot{x}_i = x_i \left(1 - x_i - \sum_{j \neq i} \alpha_{ij} x_j \right)$$

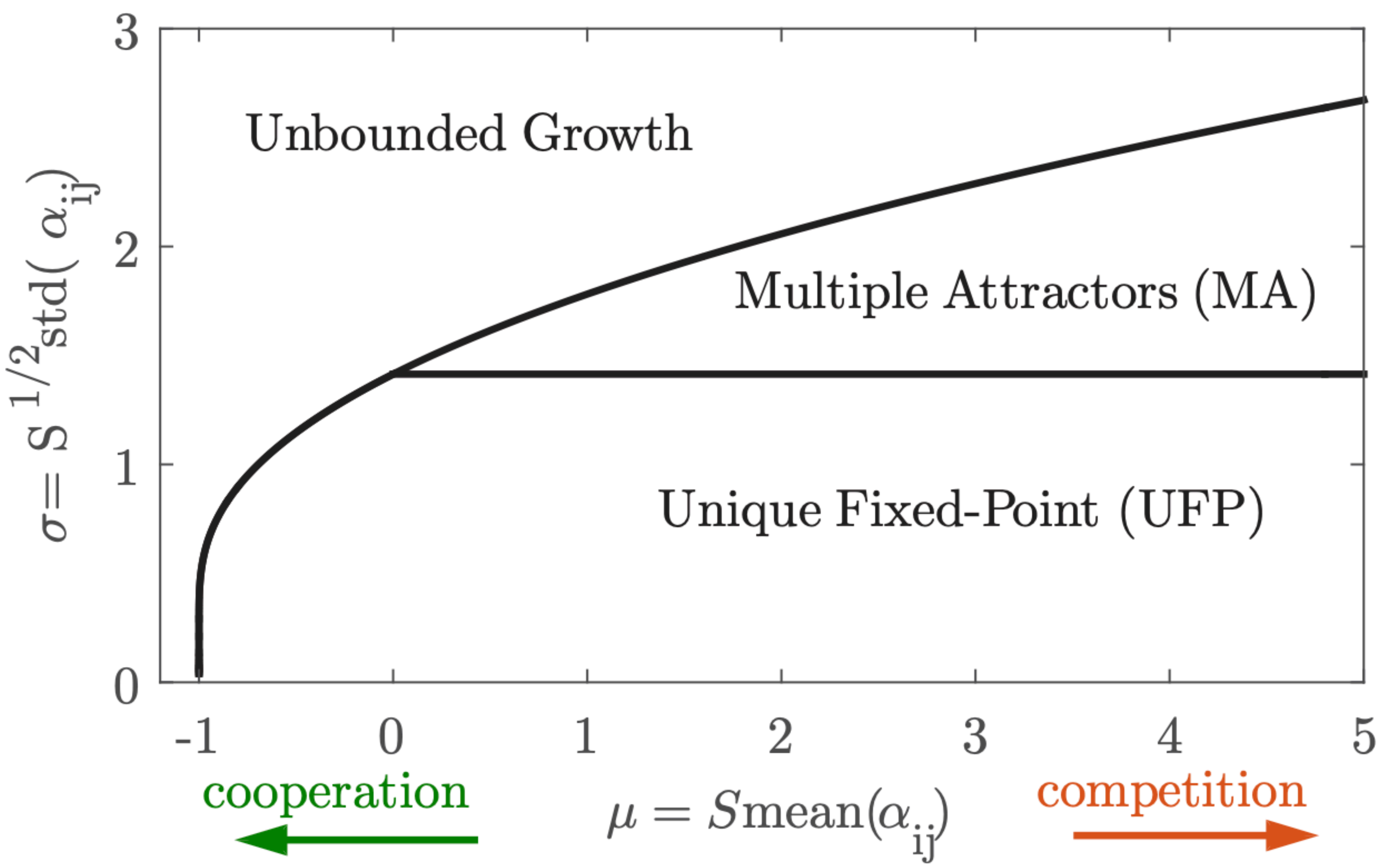
- x_i : number of individuals of species $i = 1, \ldots, N$
- With N species, $\sim N^2$ interaction parameters
- Popular approach: disordered interactions

Background: Lotka-Volterra equations

$$\dot{x}_i = x_i \left(1 - x_i - \sum_{j \neq i} \alpha_{ij} x_j \right)$$

$$\text{mean}(\alpha_{ij}) = \mu / N$$

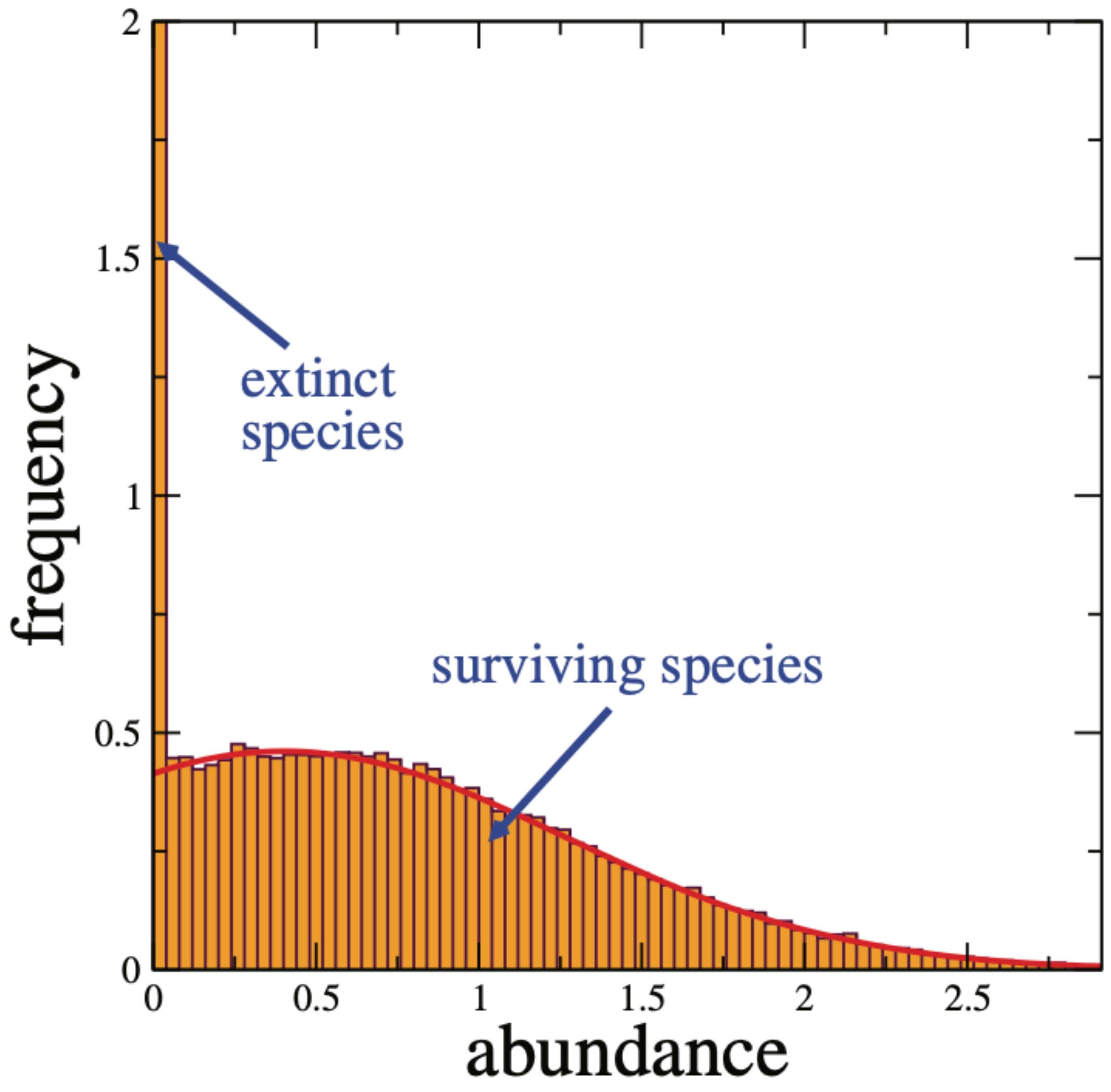
$$\text{var}(\alpha_{ij}) = \sigma^2 / N$$



[Bunin, PRE 2017]

Background: Lotka-Volterra equations

- Predicted equilibrium distribution is Gaussian
- But not observed experimentally!
  Instead power-law, gamma, log-normal, …
- Lotka-Volterra equations are reasonable.
  Discrepancy might lie in assumptions on interactions:
  - random
  - two-body
  - Gaussian
  - fixed in time (“quenched”)
  - instantaneous



[Galla, EPL 2018]

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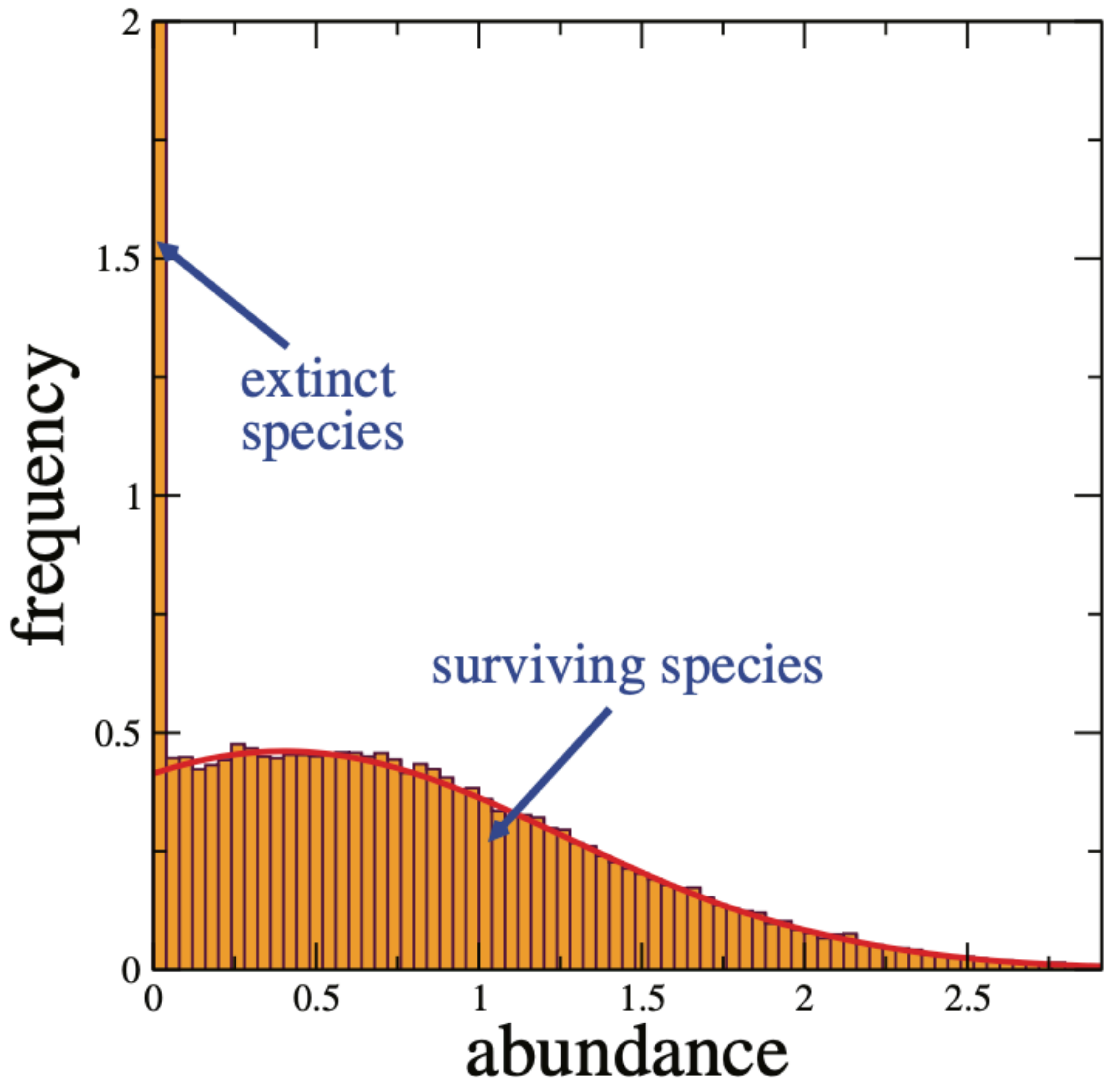


[Galla, EPL 2018]

“Annealed” interactions

- Interactions as Ornstein-Uhlenbeck processes

$$\alpha_{ij}(t) = \frac{\mu}{N} + \frac{\sigma}{\sqrt{N}} z_{ij}(t)$$

$$\langle z_{ij}(t) \rangle = 0$$

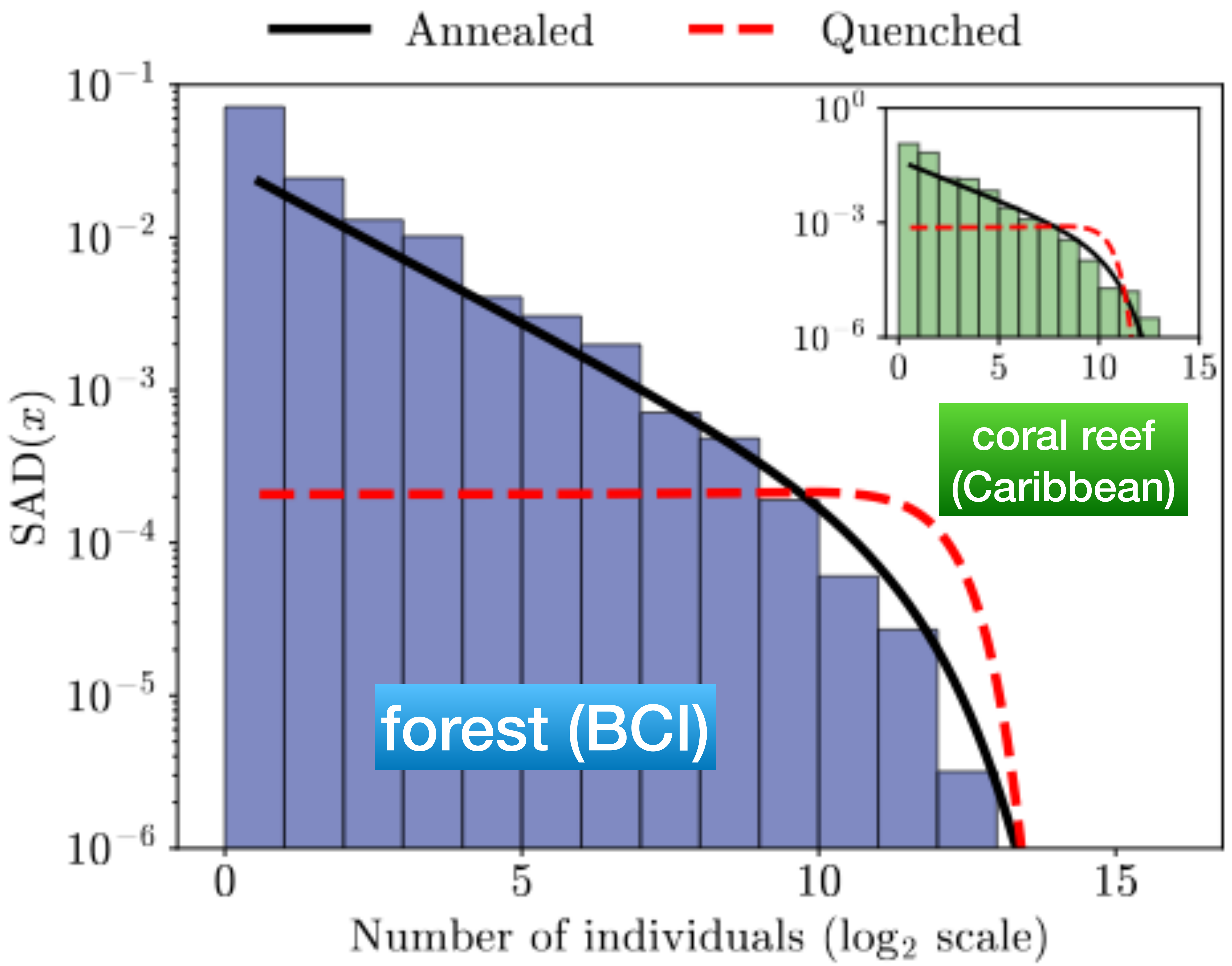
$$\langle z_{ij}(t) z_{ij}(t') \rangle = Q(t - t')$$

- Correlation time: τ
- Amplitude is chosen to have limits
  - $\tau \to \infty$: quenched disorder
  - $\tau \to 0$: white noise

$$Q(\Delta t) = \frac{1 + 2\tau}{2\tau} \exp(-|\Delta t|/\tau)$$

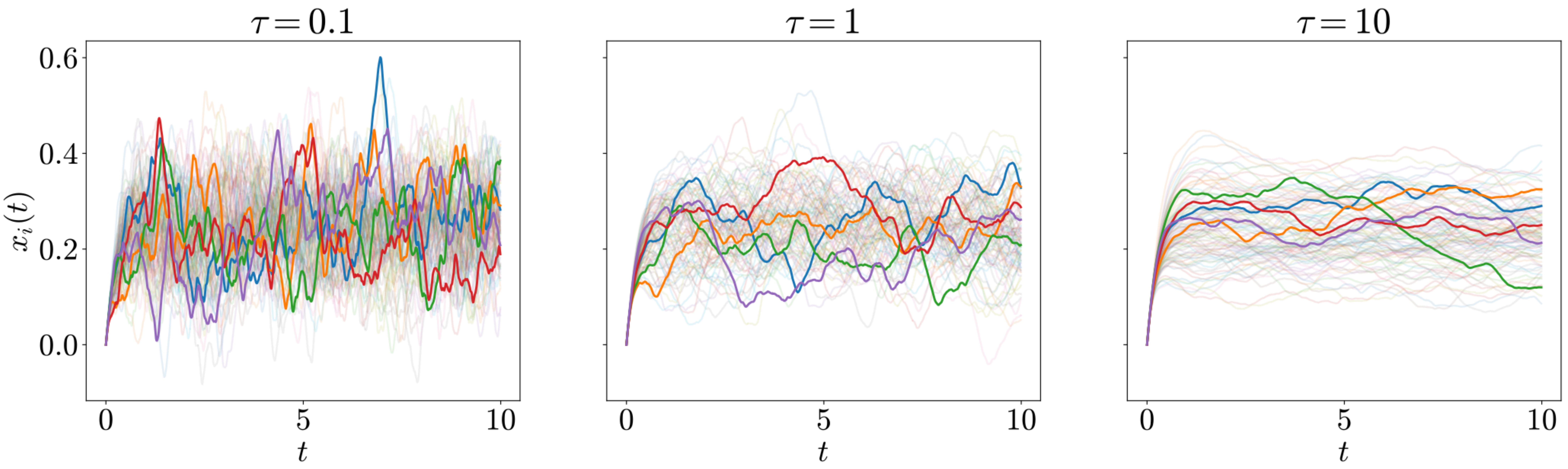
Annealed disorder fits data better!

$$\dot{x}_i = x_i \left(1 - x_i - \sum_{j \neq i} \alpha_{ij}(t) x_j \right)$$



Linear system with annealed disorder

$$\dot{x}_i(t) = h - kx_i(t) + \sum_{j \neq i} \alpha_{ij}(t) x_j(t)$$



Dynamical Mean-Field Theory

$$\dot{x}_i(t) = h - kx_i(t) + \sum_{j \neq i} \alpha_{ij}(t) x_j(t)$$

- Idea:
  - all degrees of freedom are statistically equivalent
  - write a closed equation for a representative one
  - interaction term is random, Gaussian by CLT, time-dependent
    ⇒ Gaussian process

Generating functional

$$Z[\psi] = \int D[x]\delta(x - x^*)e^{i\psi\cdot x}$$

$$D[x] = \prod_{i,t} dx_i(t) \qquad \delta(x - x^*) = \prod_{i,t} \delta(x_i(t) - x_i^*(t)) \qquad \psi \cdot x = \sum_i \int dt \psi_i(t) x_i(t)$$

- $x_i^*(t)$ are solutions of linear system for a given realization of $\alpha_{ij}(t)$
- If $Z[\psi]$ is known, all information about $x_i^*(t)$ can be obtained taking derivatives w.r.t to ψ_i

Generating functional

$$Z[\psi] = \int D[x]\delta(x - x^*)e^{i\psi\cdot x}$$

- Take average over disorder $\alpha_{ij}(t)$
- Result is $\langle Z[\psi]\rangle = Z_{eff}^N[\psi]$ with Z_{eff} generating functional of

$$\dot{x}(t) = h - kx(t) + \eta(t)$$

where $\eta(t)$ is Gaussian noise with

$$\begin{aligned}\langle \eta(t)\rangle \; &= \mu\langle x(t)\rangle \\ \langle \eta(t)\eta(t')\rangle_c &= \sigma^2 Q(t-t')\langle x(t)x(t')\rangle\end{aligned}$$

Dynamical Mean-Field Theory

$$\dot{x}_i(t) = h - kx_i(t) + \sum_{j \neq i} \alpha_{ij}(t) x_j(t)$$

$$\Downarrow$$

$$\dot{x}(t) = h - kx(t) + \eta(t)$$

- DMFT equation is same as Ornstein-Uhlenbeck SDE
- …but noise is self-consistent! Much more complicated

DMFT process is Gaussian

$$\dot{x}(t) = h - kx(t) + \eta(t)$$

- Explicit solution is

$$x(t) = x_0 e^{-kt} + \int_0^t ds e^{-k(t-s)} \left(h + \eta(t)\right)$$

- $x(t)$ is Gaussian, since it is a linear combination of Gaussian $\eta(t)$
- So we only need to find mean $\langle x(t) \rangle$ and autocorrelation $\langle x(t) x(t') \rangle$
- Mean is trivial. For autocorrelation…

Autocorrelation: PDE

- Rewrite DMFT equation as

$$\eta(t) = \dot{x}(t) + kx(t) - h$$

- Take the product $\eta(t)\eta(t')$ and average
- Result is a PDE for the autocorrelation $C(t, t') = \langle x(t)x(t') \rangle - \langle x(t) \rangle \langle x(t') \rangle$

$$\left[\partial_t \partial_{t'} + k(\partial_t + \partial_{t'}) + k^2 - \sigma^2 Q(t - t')\right] C(t, t') = f(t, t')$$

where

$$f(t, t') = \sigma^2 Q(t - t') \langle x(t) \rangle \langle x(t') \rangle$$

Autocorrelation: Riemann method

$$\left[\partial_t \partial_{t'} + k(\partial_t + \partial_{t'}) + k^2 - \sigma^2 Q(t - t')\right] C(t, t') = f(t, t')$$

- With the transformation $C(t, t') = e^{-k(t+t')} D(t, t')$ this is equivalent to

$$\left[\partial_t \partial_{t'} - \sigma^2 Q(t - t')\right] D(t, t') = e^{k(t+t')} f(t, t'),$$

- Linear hyperbolic PDE: Riemann method
  Find Riemann function $A(s, s'; t, t')$ which solves

$$\begin{cases} \left[\partial_s \partial_{s'} - \sigma^2 Q(s - s')\right] A(s, s'; t, t') = 0 \\ A(s, t'; t, t') = 1 \\ A(t, s'; t, t') = 1 \end{cases}$$

Autocorrelation: Riemann function

- Make change of variables $u = e^{-s/\tau}, u' = e^{s'/\tau}$
- With some manipulations Riemann function is

$$A(s, s'; t, t') = \begin{cases} A_0(s, s'; t, t') & s - s' > 0 \\ A_0(-s, -s'; -t, -t') & s - s' < 0 \end{cases}$$

with

$$A_0(s, s'; t, t') = J_0\left(\lambda\sqrt{(e^{-s/\tau} - e^{-t/\tau})(e^{s'/\tau} - e^{t'/\tau})}\right)$$

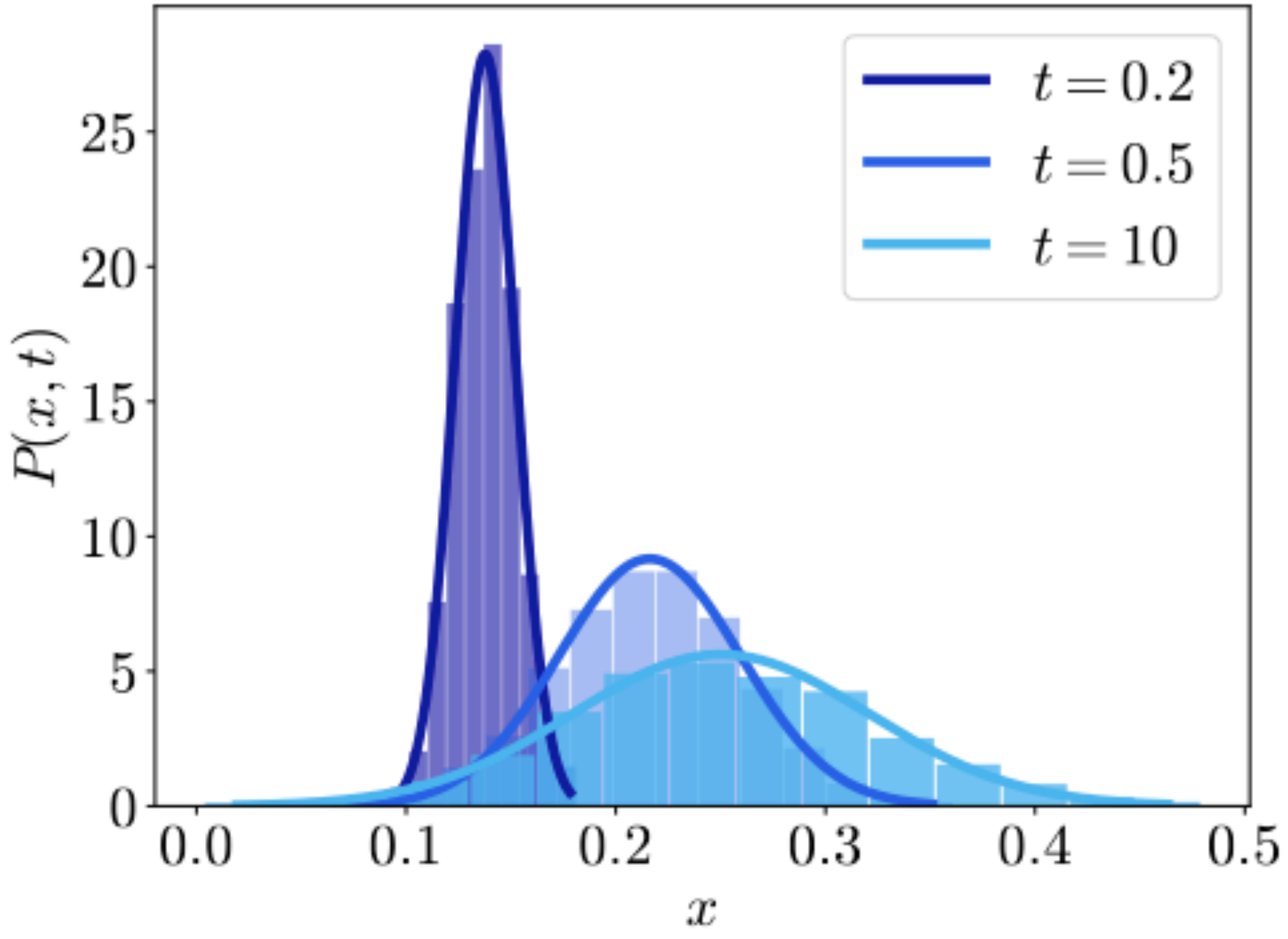
Autocorrelation: exact solution

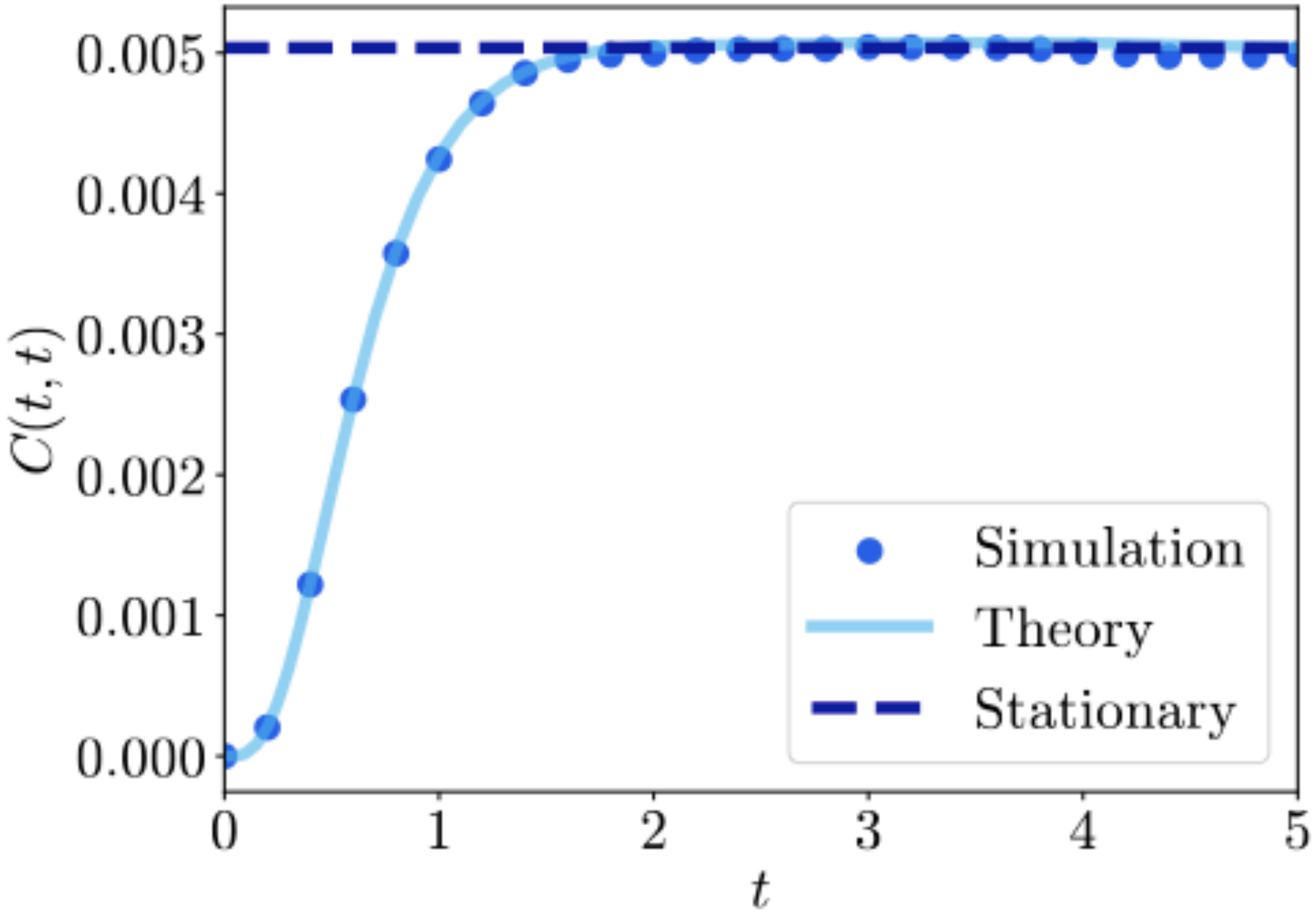
- Riemann method gives the autocorrelation as

$$C(t,t') = e^{-k(t+t')} \int_0^t ds \int_0^{t'} ds' e^{k(s+s')} A(s,s';t,t') f(s,s')$$

- We were not able to compute the integral…

Autocorrelation: numerics





Stationary autocorrelation

- At long times DMFT process becomes stationary
- Autocorrelation depends only on difference of times: $C(t, t') = C_{st}(t - t')$
- PDE becomes ODE

$$-C''_{st}(t) + \left[k^2 - \sigma^2 Q(t)\right] C_{st}(t) = \sigma^2 Q(t)\langle x\rangle^2_{st},$$

which can be solved with standard methods

Stationary variance

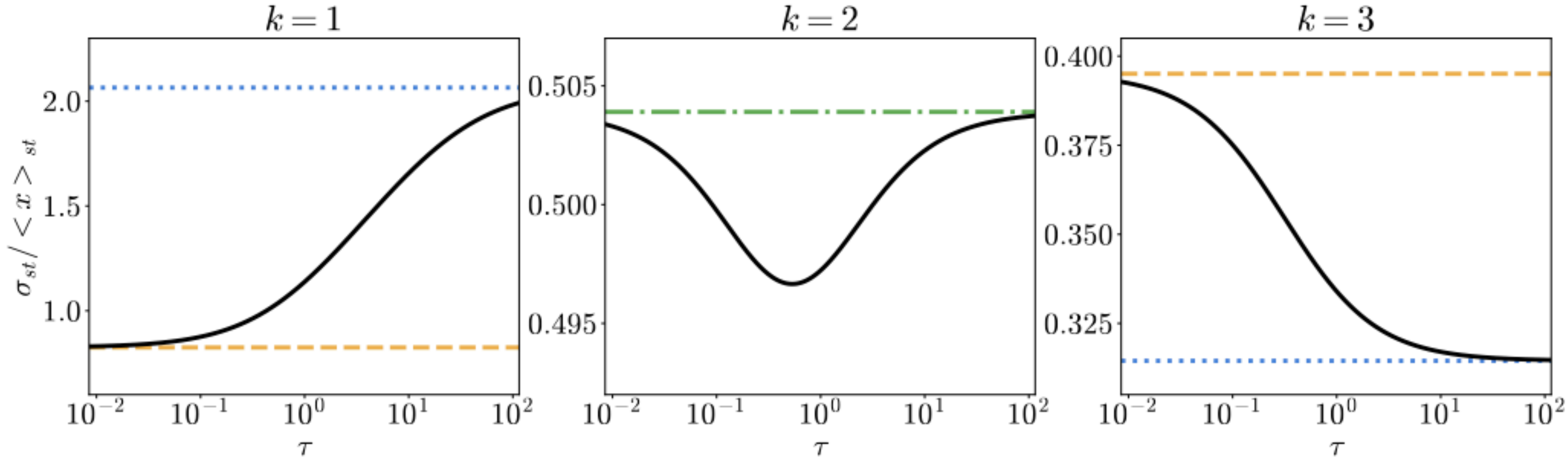
- Since $C_{st}(t) = \langle x(t)x(0)\rangle - \langle x(t)\rangle\langle x(0)\rangle$, stationary variance is $C_{st}(0)$
- It turns out that

$$\sigma_{st}^2 = \left[\frac{{}_1F_2\left(n; n+1, 2n+1; -\lambda^2/4\right)}{2\,{}_0F_1\left(2n; -\lambda^2/4\right) - {}_0F_1\left(2n+1; -\lambda^2/4\right)} - 1\right]\langle x\rangle_{st}^2,$$

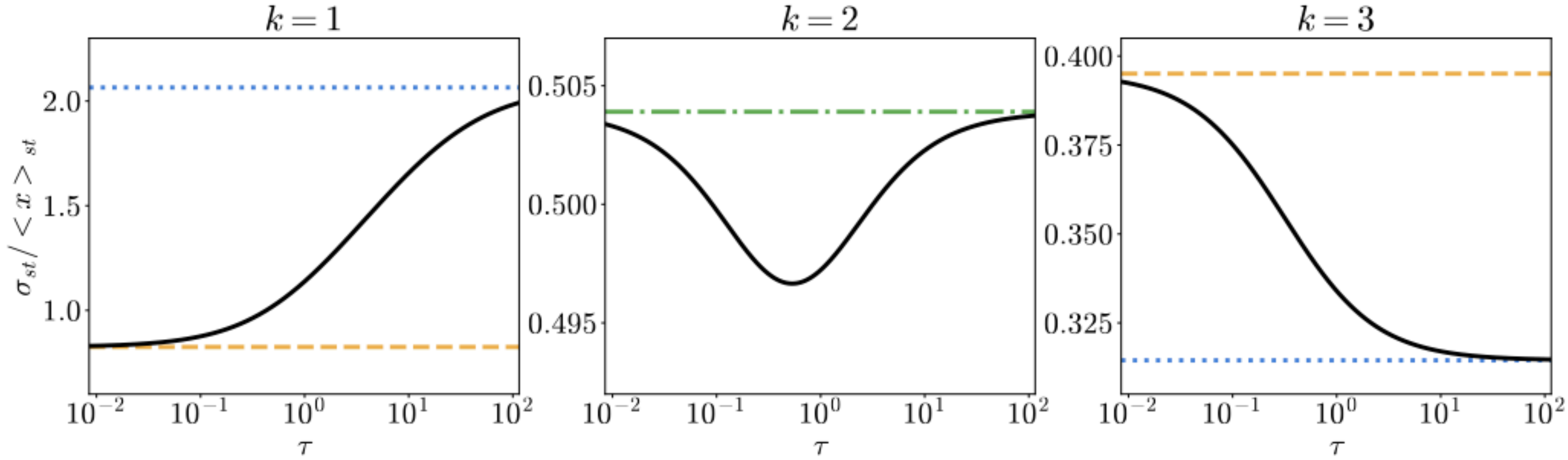
with

$$n = k\tau \qquad \lambda = \sqrt{2\tau(1+2\tau)}\sigma$$

Stationary variance

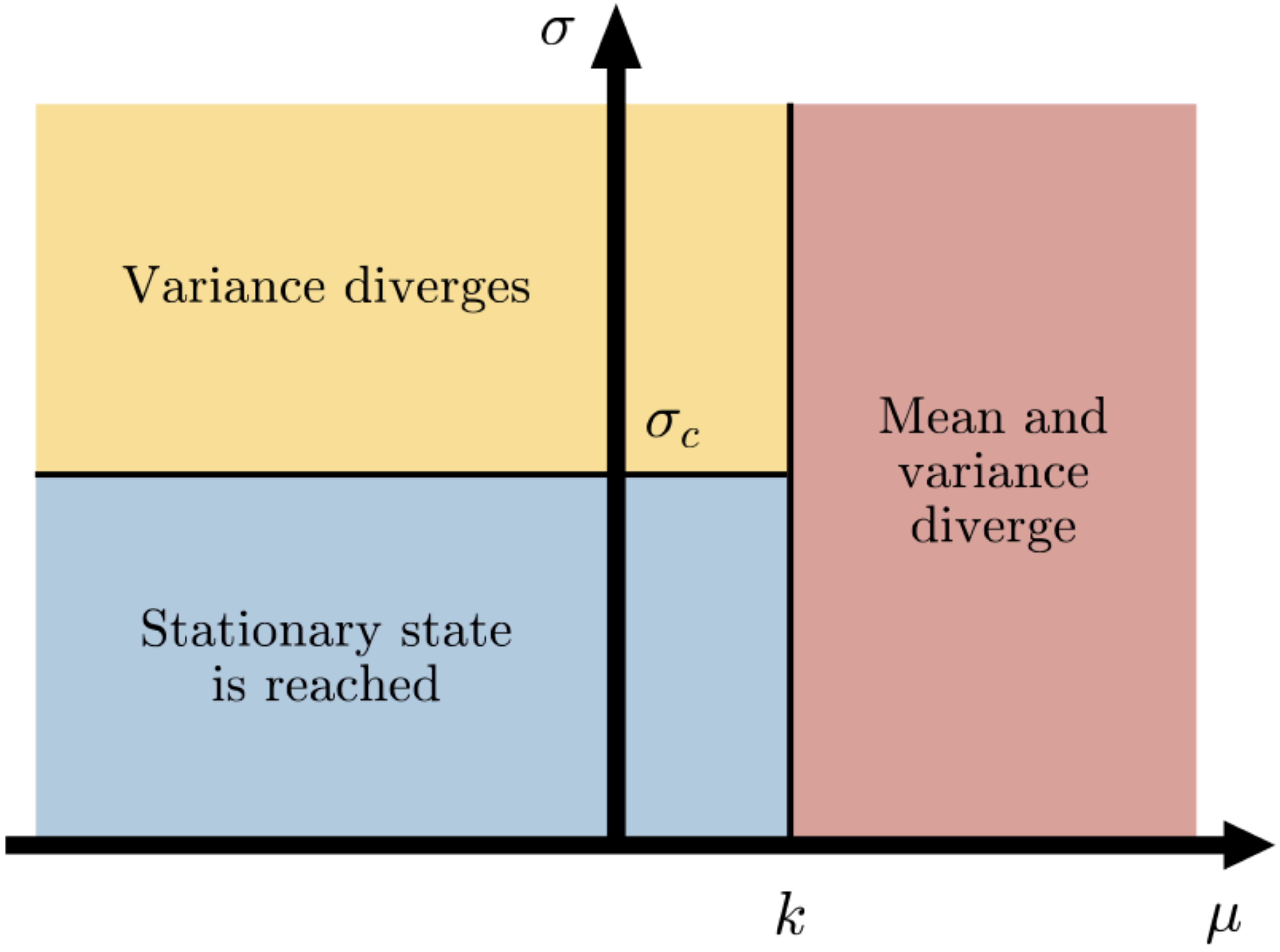


Stationary variance



non-monotonic in general!

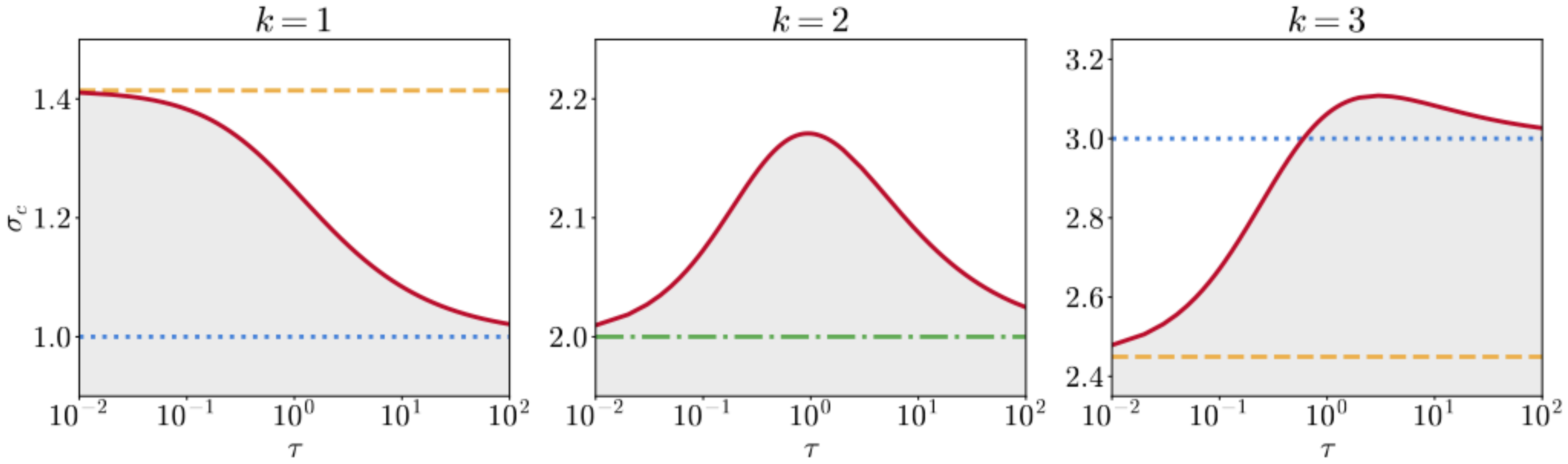
Phase diagram



Critical variance

$$2nJ_{2n}(\lambda_c) - \lambda_c J_{2n-1}(\lambda_c) = 0$$

$$n = k\tau \qquad \lambda_c = \sqrt{2\tau(1+2\tau)}\sigma_c$$



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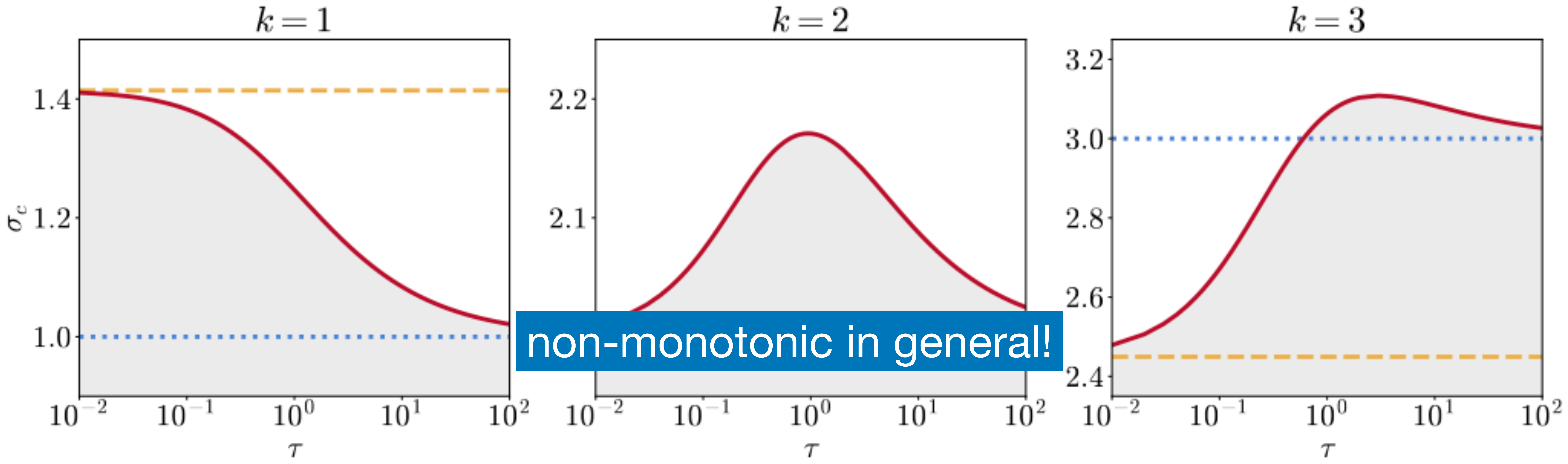

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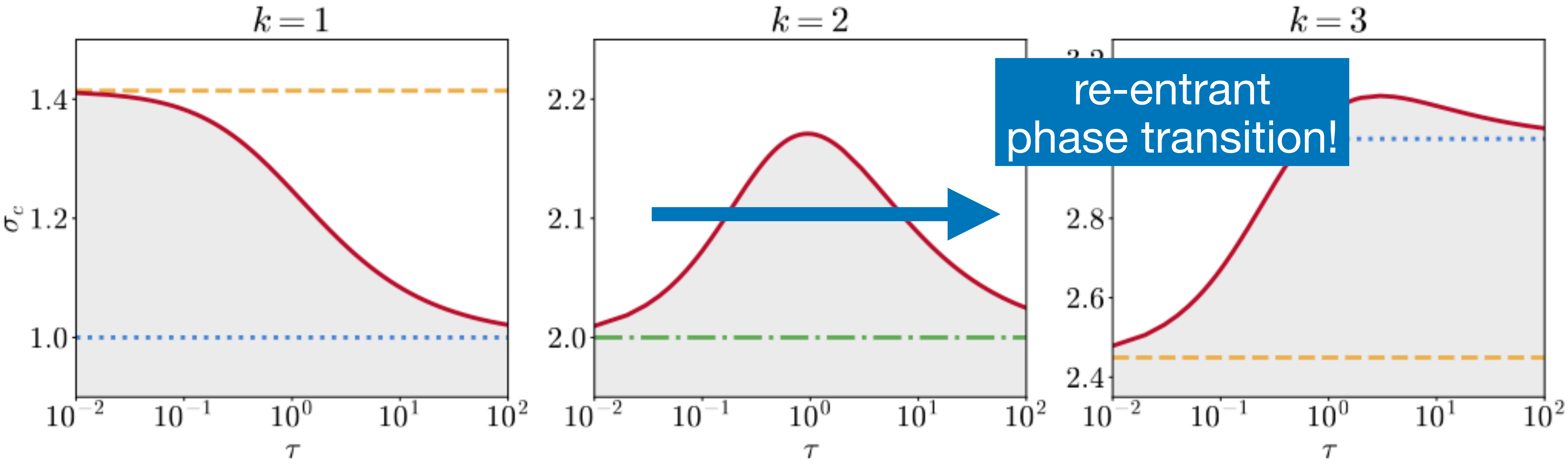

Conclusions

- We studied a multidimensional linear system with colored-noise interactions
- Using the generating functional we derived equation for a representative d.o.f.
- We showed that the solution is a Gaussian process
- We derived and solved equations for the mean and autocorrelation, solving exactly the system
- We found some unexpected features!
  - non-monotonic variance
  - re-entrant phase transition

Perspectives

- Possible extensions
  - hierarchical structure in interactions
  - finite connectivity
  - non-Gaussian interactions
- What happens with a combination of quenched and annealed disorder?
- Can the framework of annealed disorder be used to extend May’s result on stability of complex systems to time-dependent equilibria?

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