Extensions of Dynamical Mean-Field Theory and applications

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Quick intro to DMFT

- Examples of many-body dynamics
  - Generalized Lotka-Volterra equations $$\dot{N}_i = N_i \left[1 - N_i + \sum_{j \neq i} \alpha_{ij} N_j \right]$$
  - Replicator equation $$\dot{x}_i = x_i \left[f_i - \sum_j x_j f_j \right] \qquad f_i = \sum_{j \neq i} (u - w_{ij}) x_j$$
  - Firing-rate models $$\dot{h}_i = -h_i + \sum_{j \neq i} J_{ij} \tanh(g h_j)$$
- With many degrees of freedom: random interaction parameters
- Interaction term is random, Gaussian by CLT, time-dependent
  ⇒ Gaussian process

$$\sum_j \alpha_{ij} N_j(t) \quad \longrightarrow \quad \eta(t)$$

Generalized Lotka-Volterra equations

$$\dot{x}_i = x_i \left[1 - x_i + \sum_j \alpha_{ij} x_j \right]$$

$$\langle \alpha_{ij} \rangle = \mu / N$$

$$\mathrm{Var}(\alpha_{ij}) = \sigma^2 / N$$

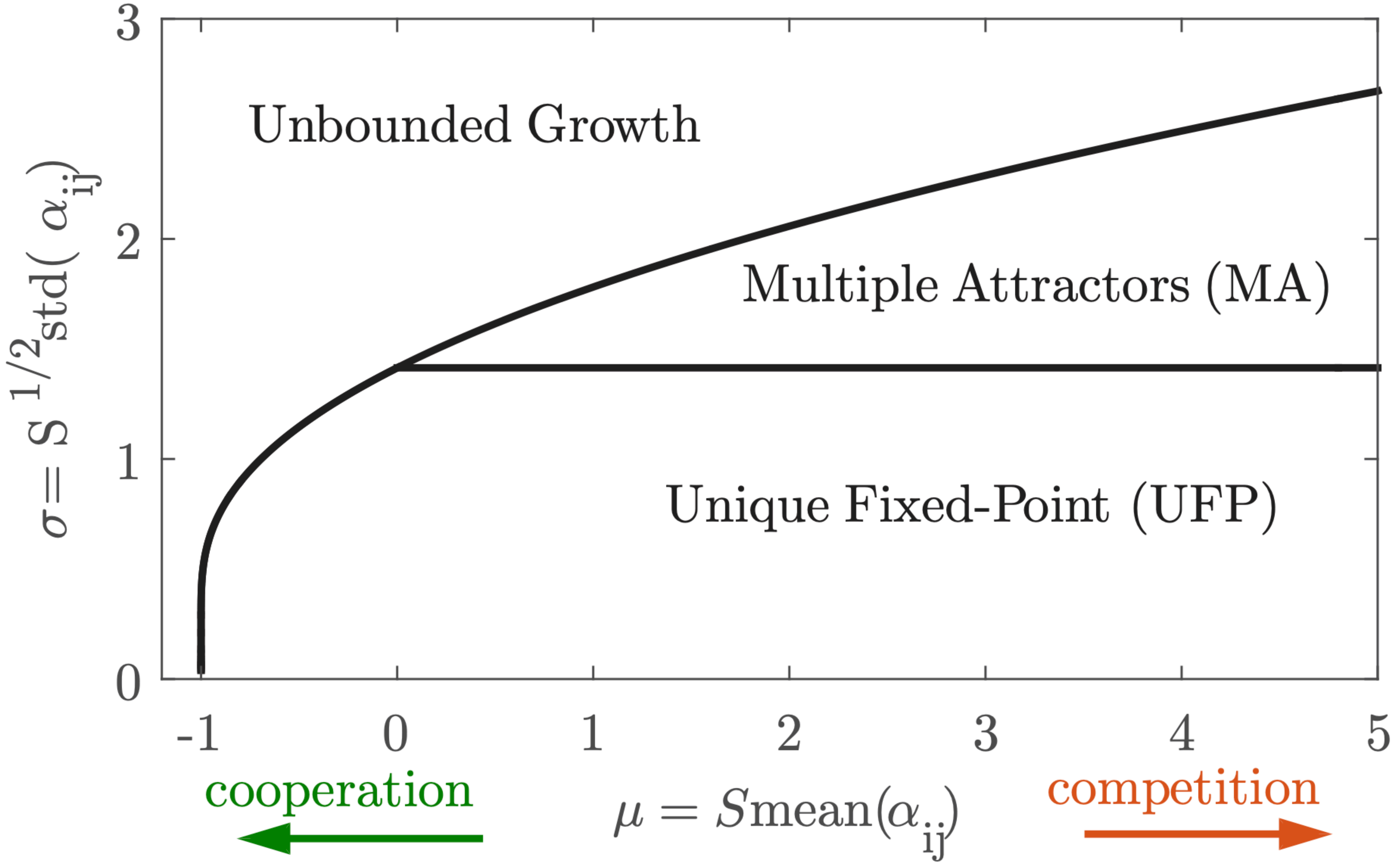
- DMFT equation is

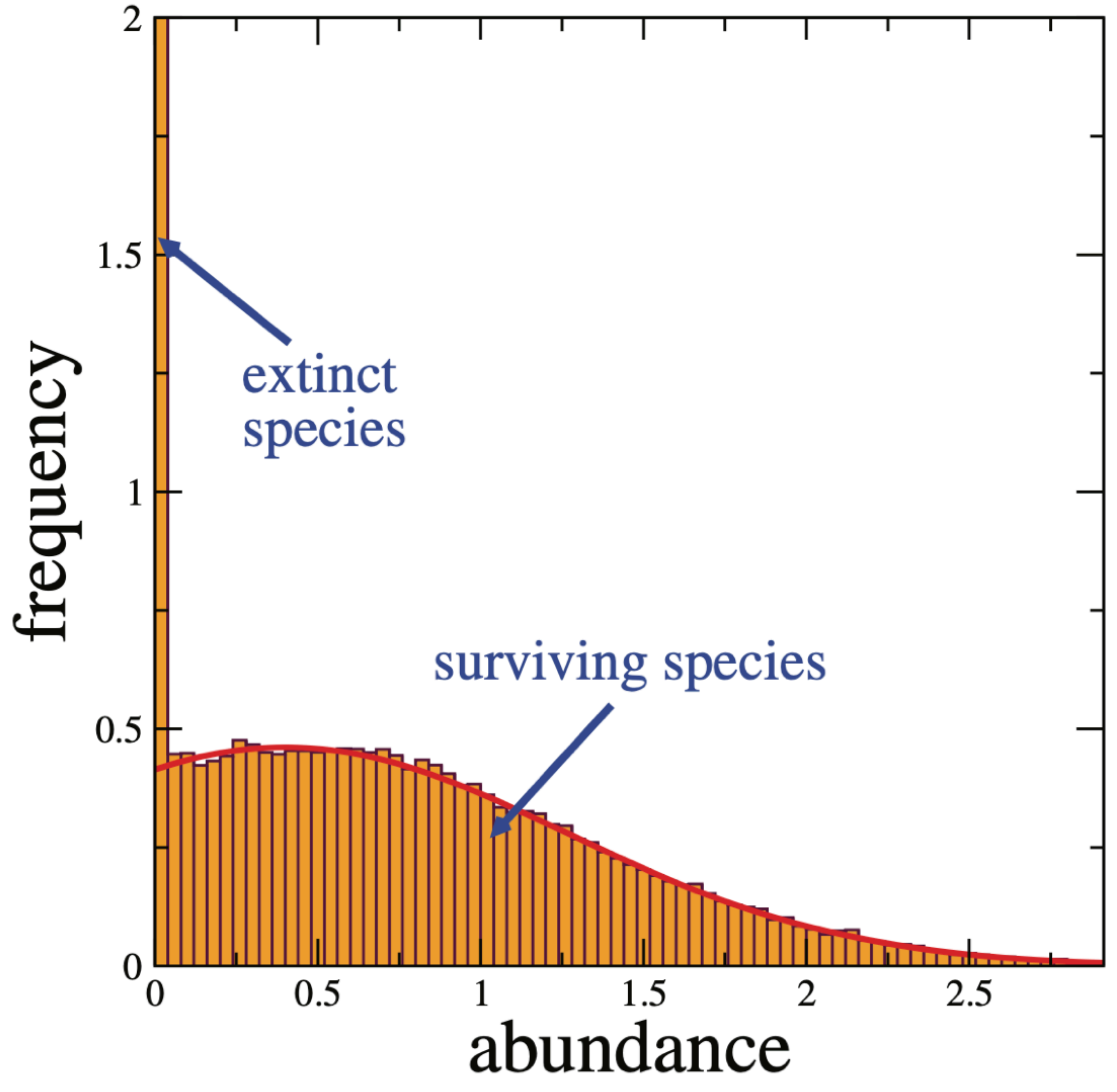
$$\dot{x} = x \left[1 - x + \eta \right]$$

$$\langle \eta(t) \rangle = \mu \langle x(t) \rangle$$

$$\langle \eta(t) \eta(t') \rangle_c = \sigma^2 \langle x(t) x(t') \rangle$$

Phase diagram and SAD





[Bunin PRE 2017, Galla EPL 2018]

Limitations of random GLV

- Predicted SAD seems universal but not observed experimentally
- Presence of unbounded growth is unphysical
- GLV assumptions:
  - interactions are fixed in time (quenched)
  - interactions are instantaneous
  - interactions are Gaussian
  - growth response is linear

GLV with time-dependent interactions

$$\dot{x}_i = x_i \left[1 - x_i + \sum_j \alpha_{ij}(t) x_j \right]$$

- Interactions are modeled as colored noise

$$\alpha_{ij}(t) = \frac{\mu}{N} + \frac{\sigma}{\sqrt{N}} z_{ij}(t)$$

$$\langle z_{ij}(t) \rangle = 0$$

$$\langle z_{ij}(t) z_{ij}(t') \rangle = Q(t - t') \propto e^{-|t-t'|/\tau}$$

- DMFT is

$$\dot{x} = x \left[1 - x + \eta \right]$$

$$\langle \eta(t) \rangle = \mu \langle x(t) \rangle$$

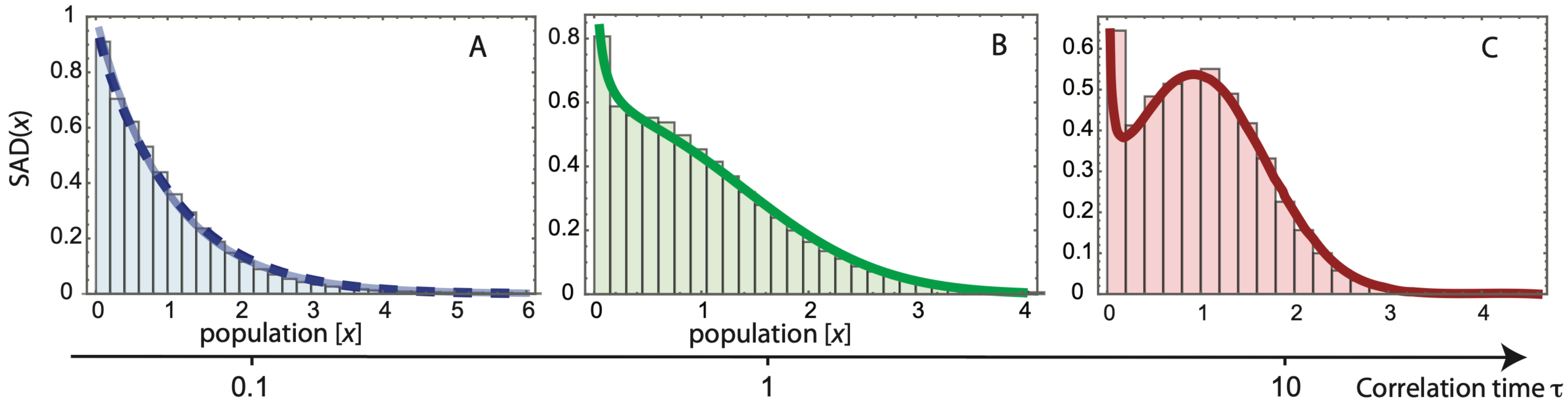
$$\langle \eta(t) \eta(t') \rangle_c = \sigma^2 Q(t - t') \langle x(t) x(t') \rangle$$

[Suweis et al. - arXiv:2307.02851]

GLV with time-dependent interactions

- SAD is interpolation between Gamma ($\tau = 0$) and Gaussian ($\tau = \infty$)

$$\mathrm{SAD}(x) \propto \left(\frac{1}{\bar{\tau}} + x\right) x^{\delta-1} e^{-x/D} \cdot e^{-\bar{\tau}(x-\bar{x})^2/2D}$$



[Suweis et al. - arXiv:2307.02851]

Linear system with colored-noise interactions

$$\dot{x}_i(t) = h - kx_i(t) + \sum_j \alpha_{ij}(t)x_j(t)$$

- Distribution of x_i at stationarity is Gaussian with

$$\frac{\sigma_{st}^2}{\langle x \rangle^2} = \frac{{}_1F_2\left(n; n+1, 2n+1; -\lambda^2/4\right)}{2\,{}_0F_1\left(2n; -\lambda^2/4\right) - {}_0F_1\left(2n+1; -\lambda^2/4\right)} - 1$$

$$n = k\tau \qquad \lambda = [2\tau(1+2\tau)]^{1/2}\sigma$$

- Unexpected features emerge! Check out Christian's poster…
  e.g: variance of x_i increases or decreases with correlation time of noise?

[Ferraro et al. - arXiv:2405.05183]

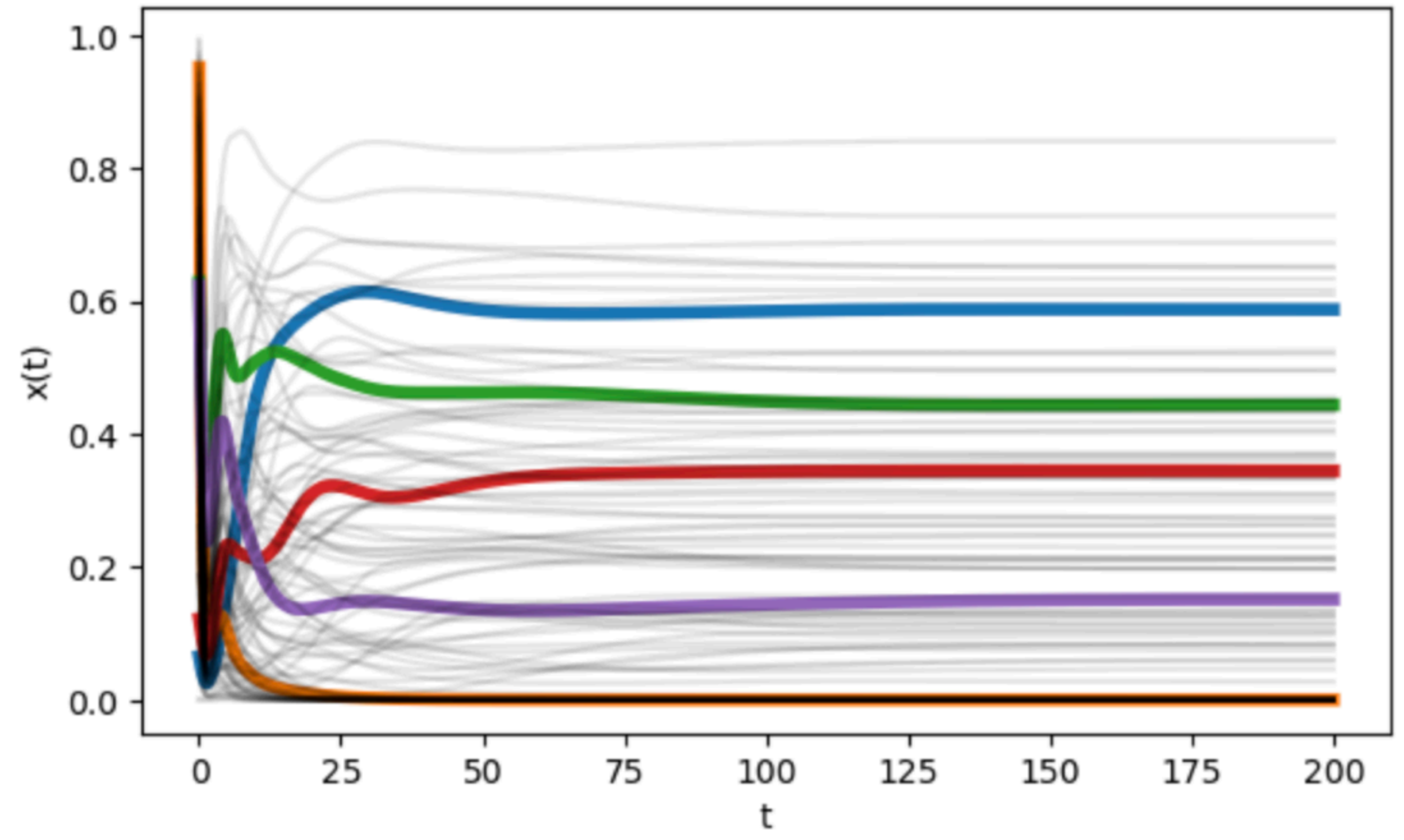
GLV with delayed interactions

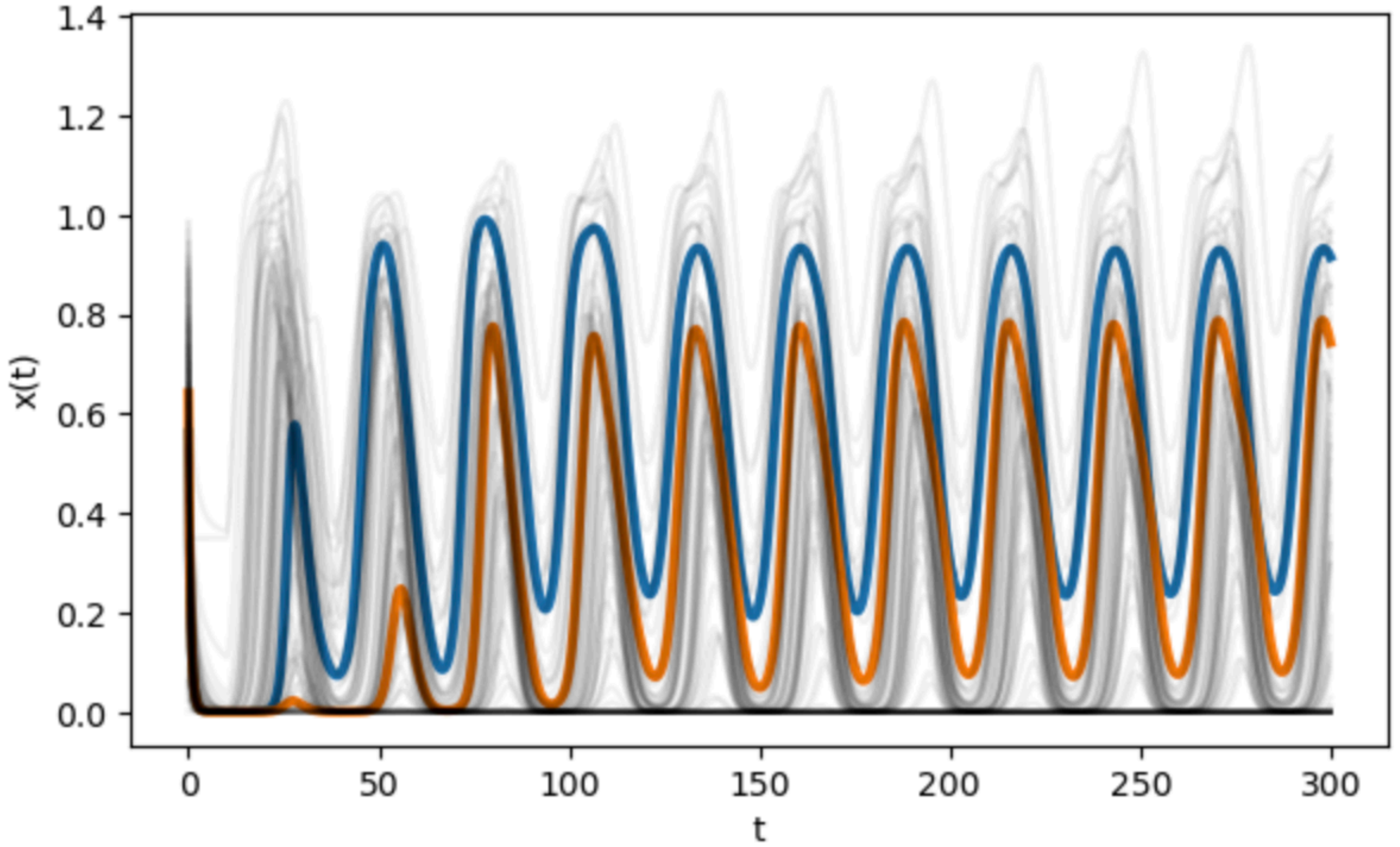
$$\dot{x}_i(t) = x_i(t)\left[1 - x_i(t-\tau) + \sum_{j\neq i} \alpha_{ij} x_j(t-\tau)\right]$$

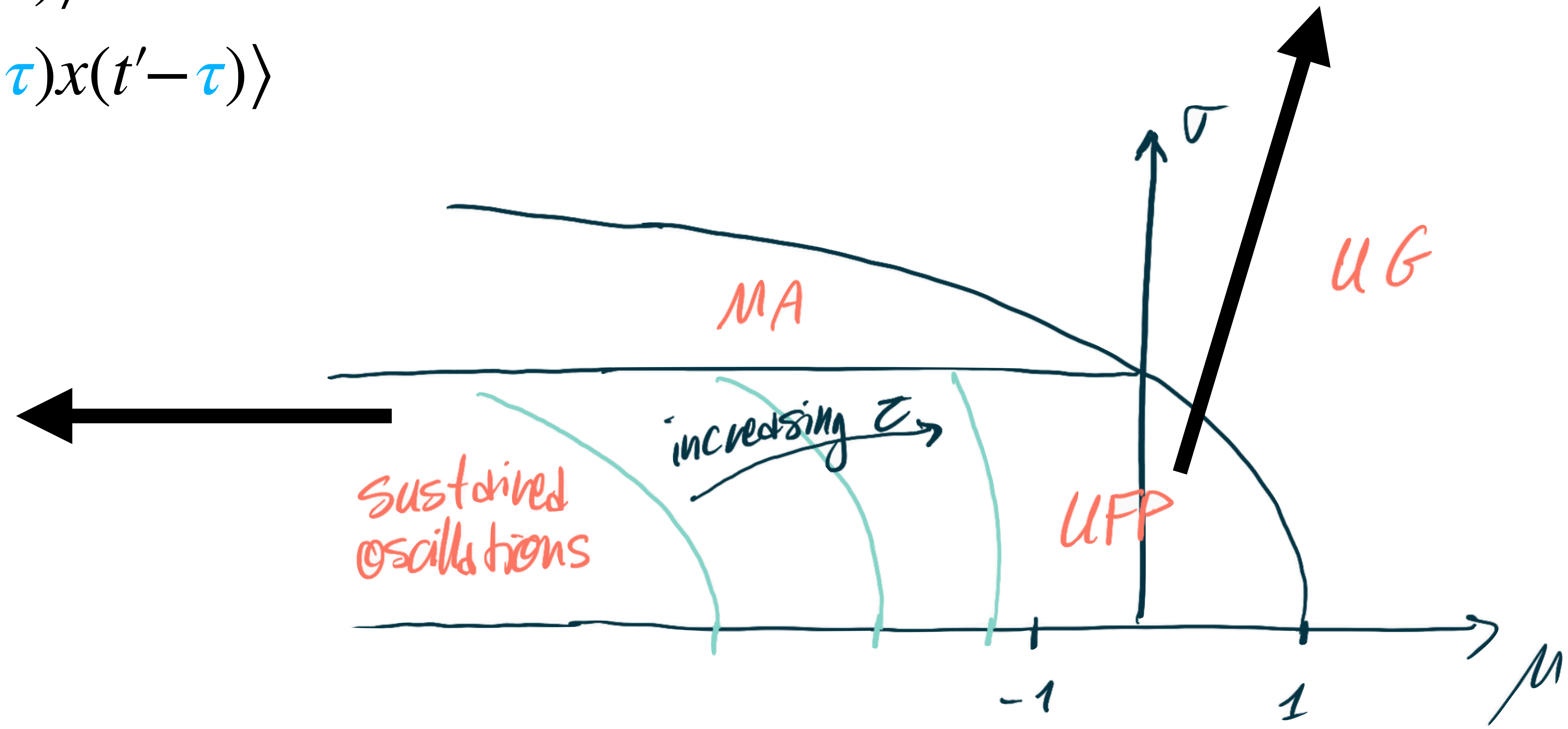
$$\dot{x}(t) = x(t)\left[1 - x(t-\tau) + \eta(t)\right]$$

$$\langle \eta(t) \rangle = \langle x(t-\tau) \rangle$$

$$\langle \eta(t)\eta(t') \rangle_c = \langle x(t-\tau)x(t'-\tau) \rangle$$







GLV with non-Gaussian interactions

- Usual assumption is only two cumulants dominate at large N

$$\lim_{N\to\infty} N \log\langle e^{iz\alpha}\rangle_\alpha = i\mu z - \frac{1}{2}\sigma^2 z^2$$

- What if all cumulants are of order N^{-1}?

$$\lim_{N\to\infty} N \log\langle e^{i\alpha z}\rangle_\alpha = F(z)$$

$$N^* = \max\left(0, 1+\eta\right) \qquad P(\eta) = \int \frac{dz}{2\pi} \exp\left[iz\eta + \int_{-1}^{\infty} d\eta' P(\eta') F(z + z\eta')\right]$$

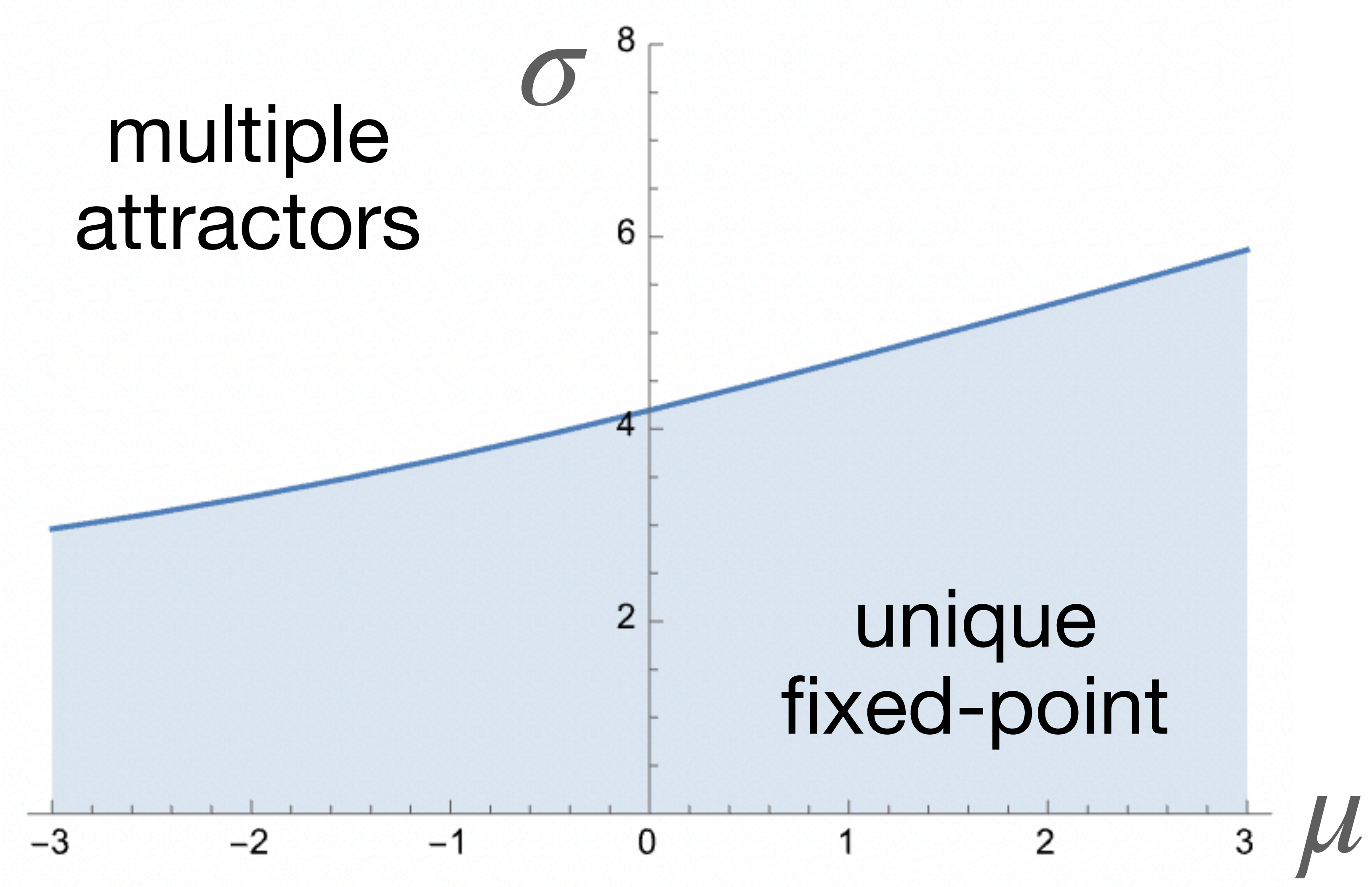
[Azaele, Maritan - arXiv:2306.13449]

GLV with saturation function

- Non-linear response function

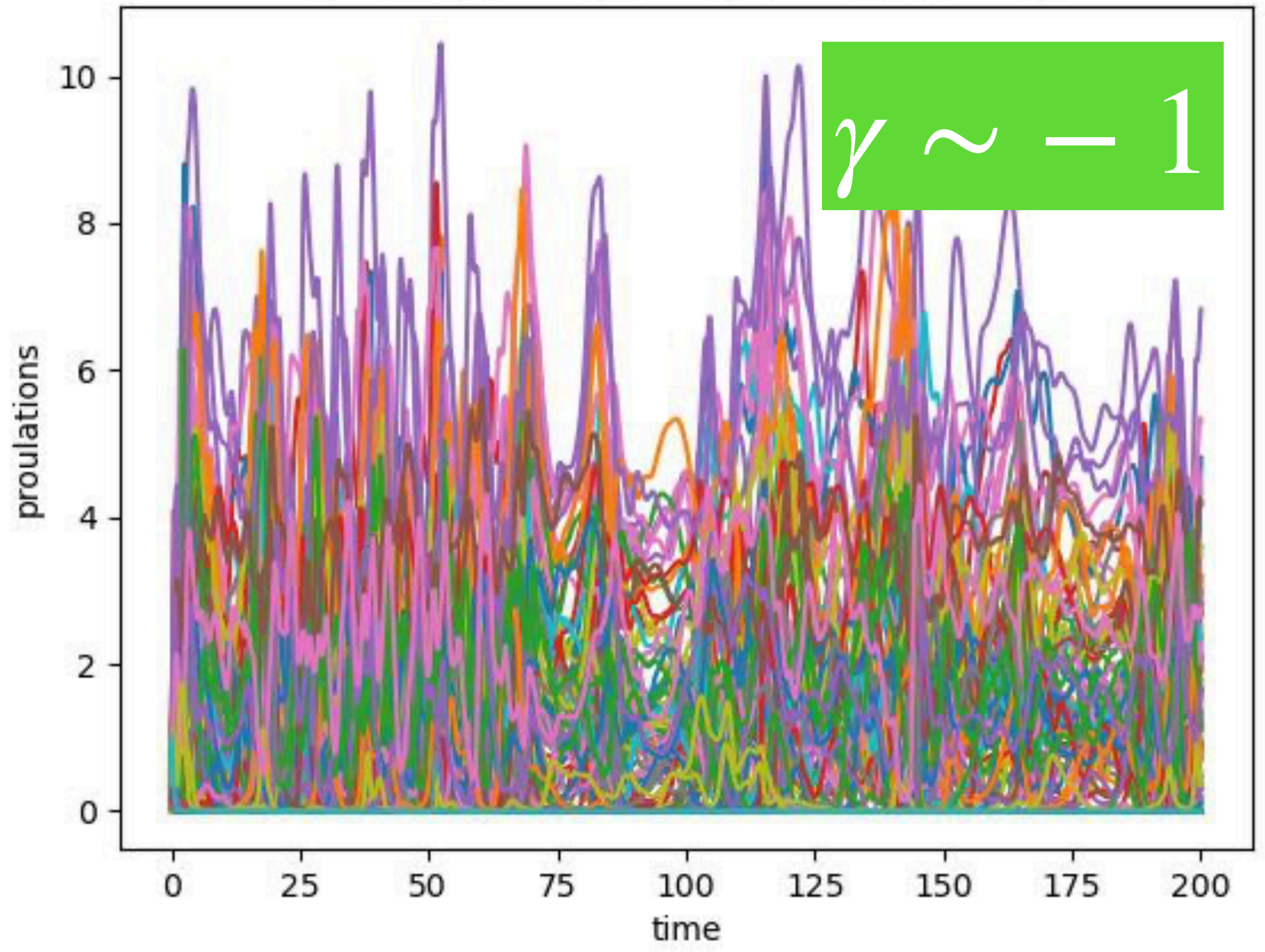
$$\dot{x}_i = x_i\left[1 - x_i + \sum_{j\neq i} \alpha_{ij}\, J(x_j)\right] \qquad J(x_j) = \frac{x_j}{K + x_j}$$

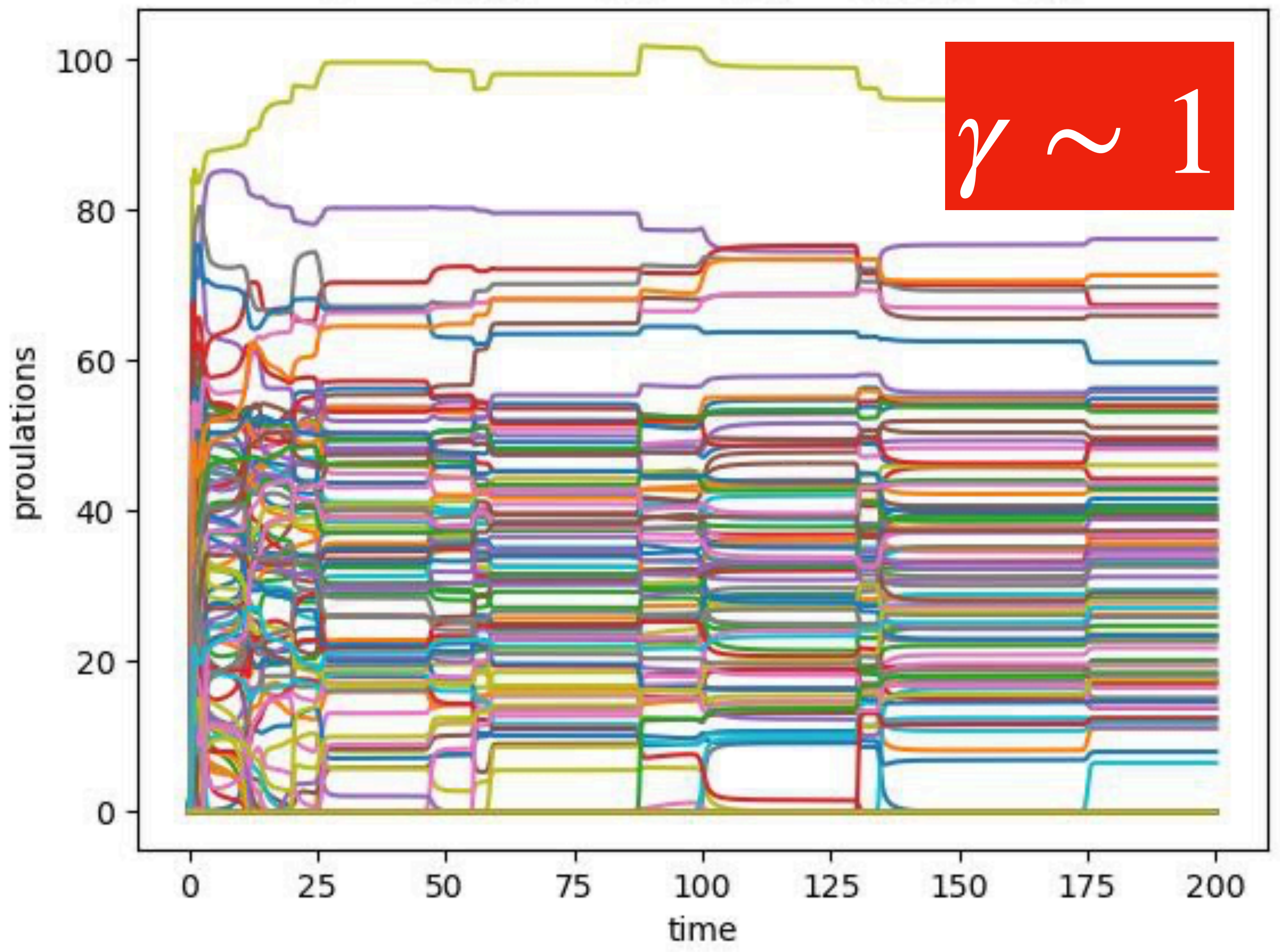
- No unbounded growth



- Two distinct chaotic regimes

$$\gamma = \mathrm{corr}(\alpha_{ij}, \alpha_{ji})$$





Conclusions

- Many possible extensions of usual DMFT are possible! Interactions can be:
  - colored-noise / “annealed”
  - delayed
  - non-Gaussian
  - non-linear
- These modifications give rise to ecologically relevant phenomena in GLV
- Framework applicable to any many-body dynamics

Thank you for your attention!