Dynamical Mean-Field Theory on random networks

Journal Club

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References

- JI Park et al, "Incorporating Heterogeneous Interactions for Ecological Biodiversity", arXiv:2403.15730
- L Poley, T Galla, JW Baron, "Interaction networks in persistent Lotka-Volterra communities", arXiv:2404.08600
- F Aguirre-López, "Heterogeneous mean-field analysis of the generalized Lotka-Volterra model on a network", arXiv:2404.11164

Quick intro to DMFT
Examples of many-body dynamics

- Generalized Lotka-Volterra equations $$\dot{N}_i = N_i \left[1 - N_i + \sum_{j \neq i} \alpha_{ij} N_j \right]$$
- Replicator equation $$\dot{x}_i = x_i \left[f_i - \sum_j x_j f_j \right] \qquad f_i = \sum_{j \neq i} (u - w_{ij}) x_j$$
- Firing-rate models $$\dot{h}_i = -h_i + \sum_{j \neq i} J_{ij} \tanh(g h_j)$$

Quick intro to DMFT

Random interaction parameters

- With N degrees of freedom $\sim N^2$ interaction parameters
  $\Rightarrow$ take them as random!

$$\dot{N}_i(t) = N_i(t)\left[1 - N_i(t) + \sum_{j \neq i} \alpha_{ij} N_j(t)\right]$$

$$\langle \alpha_{ij} \rangle = \frac{\mu}{N}$$

$$\mathrm{Var}(\alpha_{ij}) = \frac{\sigma^2}{N}$$

Quick intro to DMFT
DMFT idea

- D.o.f. are all equivalent, so we try to write effective single-d.o.f. equation
- Interaction term is random, Gaussian by CLT, time-dependent
  ⇒ Gaussian process

$$\sum_{j\neq i} \alpha_{ij} N_j(t) \quad \longrightarrow \quad \eta(t)$$

- DMFT for GLV

$$\dot{N}_i(t) = N_i(t)\left[1 - N_i(t) + \sum_{j\neq i} \alpha_{ij} N_j(t)\right] \quad \longrightarrow \quad \dot{N}(t) = N(t)\big[1 - N(t) + \eta(t)\big]$$

$$\langle \eta(t) \rangle = \mu \langle N(t) \rangle$$

$$C_\eta(t,t') = \sigma^2 \langle N(t) N(t') \rangle$$

GLV on random networks
Model definition

- DMFT is exact, but only in infinite dims (everyone interacts with everyone)
- Adding a network structure in GLV

$$\dot{N}_i(t) = N_i(t)\left[1 - N_i(t) + \sum_{j\neq i} A_{ij}\alpha_{ij}N_j(t)\right]$$

- Random configuration model with degree distribution $p(k)$

$$A_{ij} = 0,1 \qquad A_{ij} = A_{ji} \qquad \mathrm{Pr}(A_{ij} = 1) = \frac{k_i k_j}{dN}$$

GLV on random networks
Derivation of DMFT

- Quite technical, done with generating functional formalism
- Average over disorder

$$\left\langle \exp\left[iA_{ij}\left(\alpha_{ij}\chi_{ij}+\alpha_{ji}\chi_{ji}\right)\right]\right\rangle_{A,\alpha} = \Pr(A_{ij}=0)+\Pr(A_{ij}=1)\left\langle \exp\left[i\left(\alpha_{ij}\chi_{ij}+\alpha_{ji}\chi_{ji}\right)\right]\right\rangle_{\alpha}$$

- Change of variables

$$P_k[x,\hat{x}] = \frac{1}{N_k}\sum_{i\in S_k}\prod_t \delta\Big(x(t)-x_i(t)\Big)\delta\Big(\hat{x}(t)-\hat{x}_i(t)\Big)$$

GLV on random networks

DMFT equation(s)

- Species are not equivalent, they differ by degree!
- So result is one DMFT equation for each degree

$$\dot{N}_k(t) = N_k(t)\left[1 - N_k(t) + \frac{\mu k}{d^2}\sum_{k'} p_{k'}k'\langle N_{k'}(t)\rangle + \eta_k(t)\right]$$

$$\langle \eta_k(t)\eta_k(t')\rangle = \frac{\sigma^2 k}{d^2}\sum_{k'} p_{k'}k'\langle N_{k'}(t)N_{k'}(t')\rangle$$

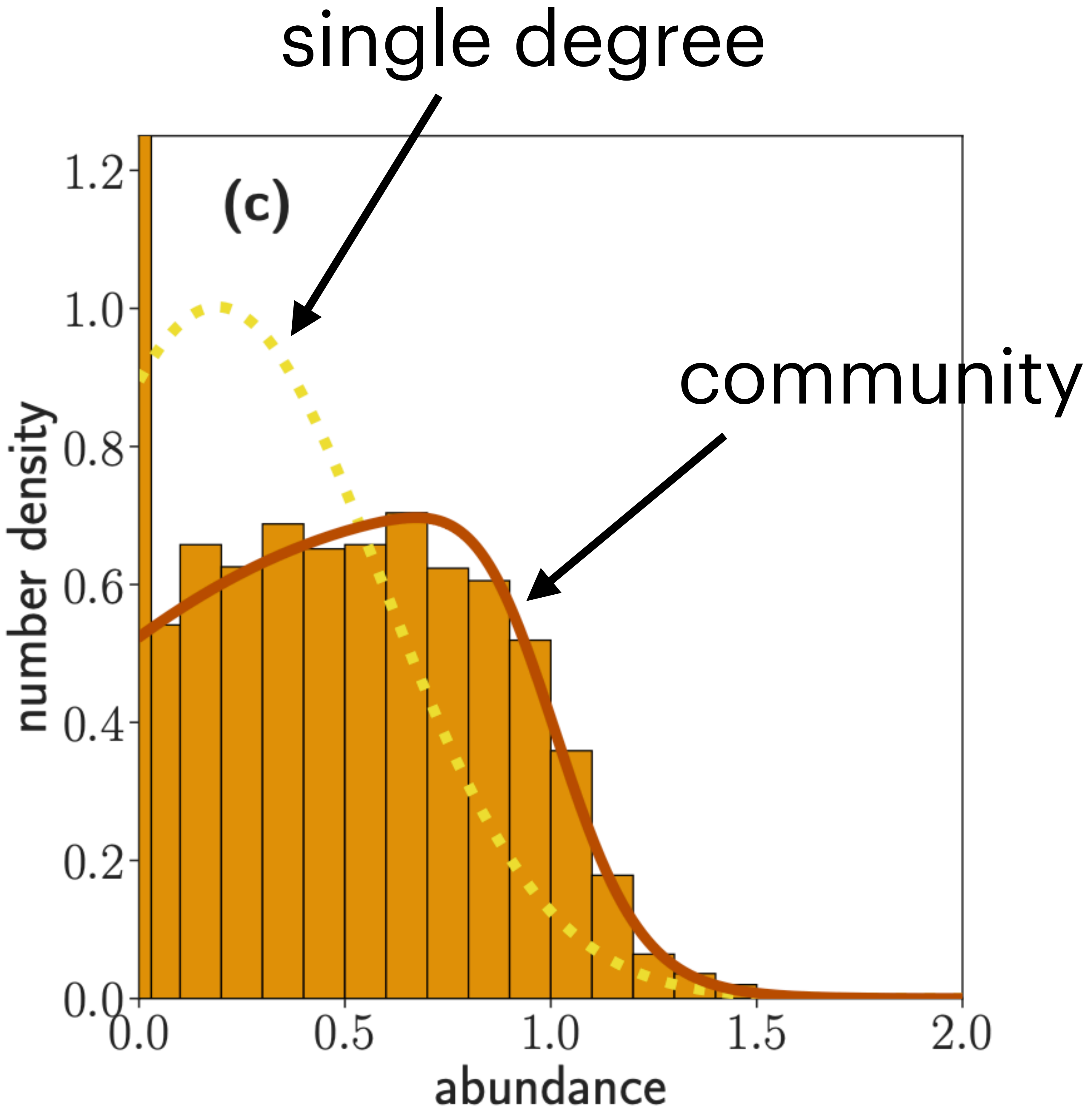
GLV on random networks

Species Abundance Distribution

- SAD of a species of degree *k*: truncated Gaussian $\text{SAD}_k(x)$
- SAD of all community:

$$\text{SAD}(x) = \sum_k p_k \text{SAD}_k(x)$$

(no explicit formula)

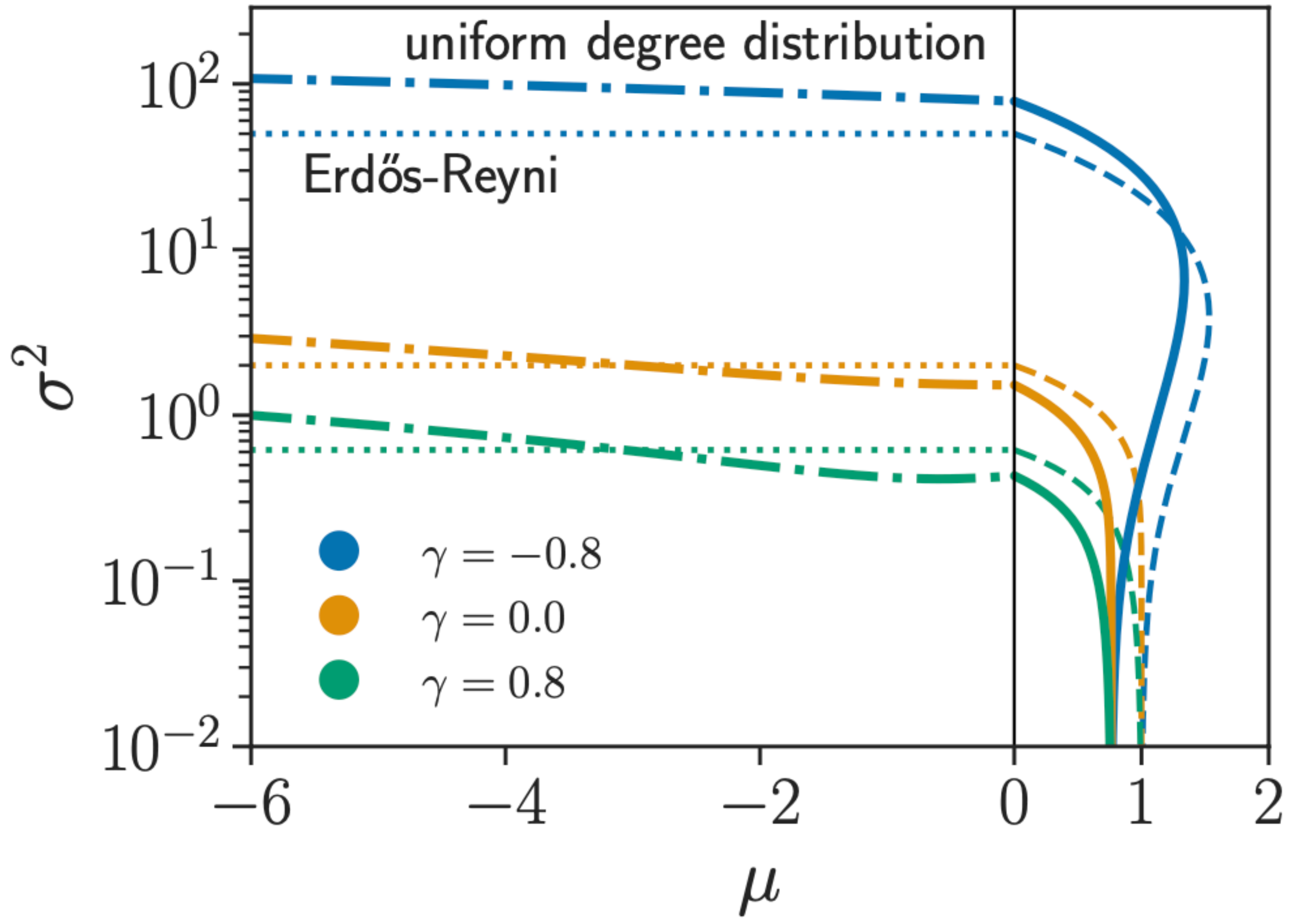


GLV on random networks
Stability of fixed point phase

- Stability condition

$$\frac{\sigma^2}{d^2}\sum_k p_k \frac{k^2}{\phi_k} < 1$$

(ϕ_k: probability of survival)



GLV on random networks
Properties of surviving community

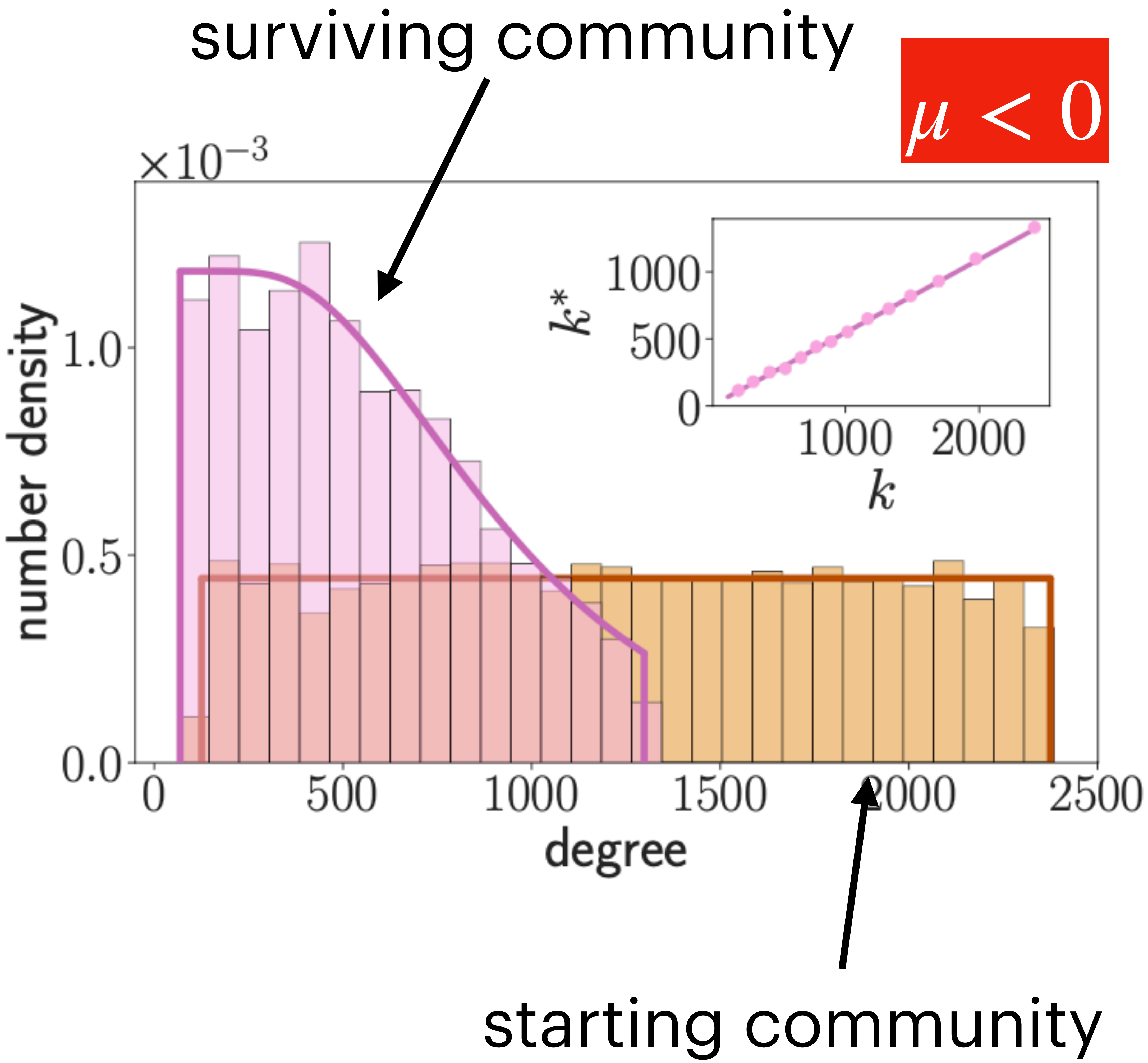
- Average degree of surviving species:

$$\langle k_i^* \rangle = r k_i$$

$$r = \frac{\sum_k p_k k \phi_k}{\sum_k p_k k}$$

- Degree distribution of surviving community:

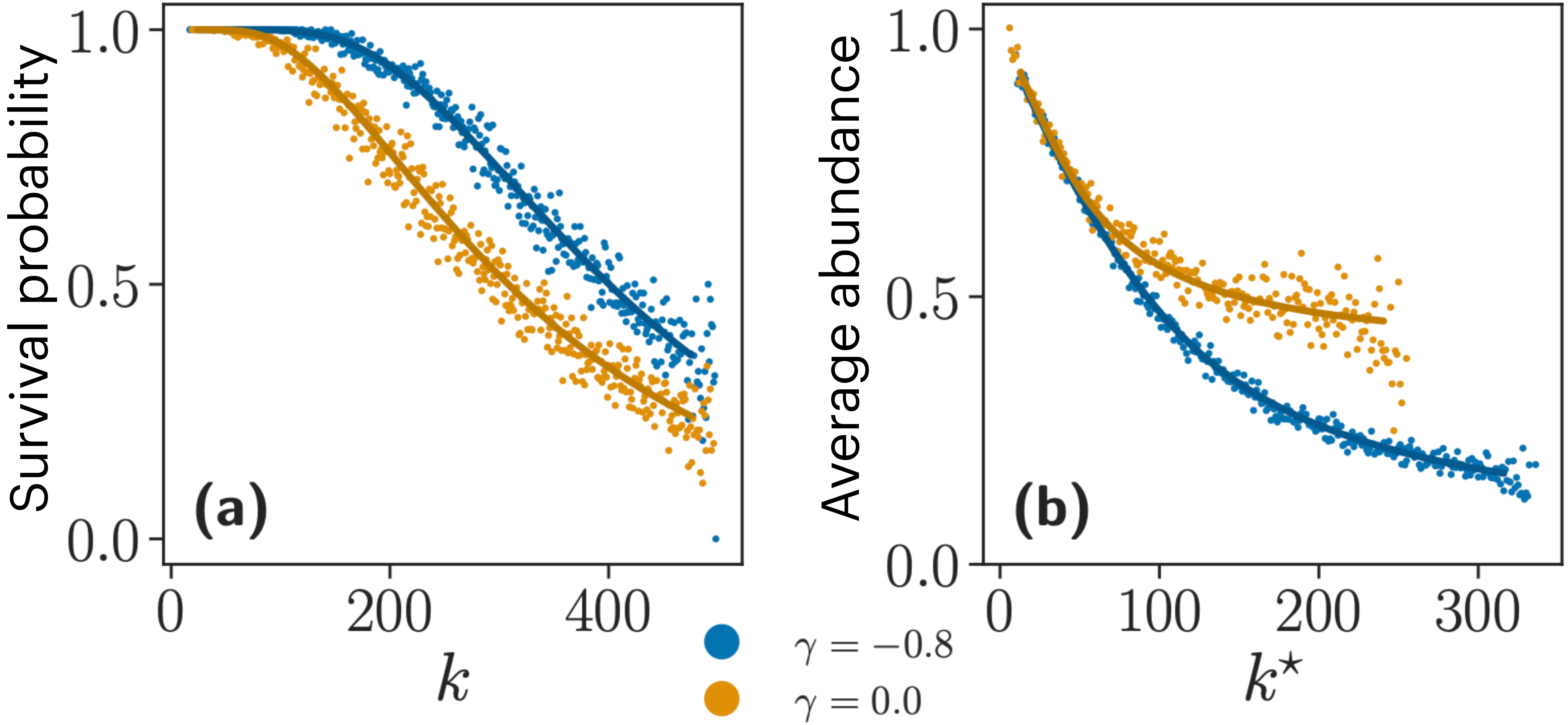
$$P^*(k^*) \propto \phi\left(\frac{k^*}{r}\right) P\left(\frac{k^*}{r}\right)$$



GLV on random networks

Properties of surviving community

$\mu < 0$



Conclusions on Poley-Galla-Baron paper

- Important advance of network structure in interactions
- Framework applicable to any many-body system
- Well written and very detailed paper; interesting technical part

- Unclear points
  - ecological implications are expected (in competitive communities)
  - results are present for dense networks i.e. $\langle k \rangle \propto N$
  - what happens to UFP when $\langle k^2 \rangle = \infty$?

Comparisons of papers

- All papers arrive at same DMFT equations, SADs, etc
- All papers consider the case of dense networks
- Poley-Galla-Baron additionally study properties of surviving community
- Aguirre-López is quite difficult to follow…

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