CCS/ITALY 2023
Naples, 9th – 11th October 2023
Across Complexity and Risk

NATIONAL BIODIVERSITY FUTURE CENTER

UNIVERSITÀ DEGLI STUDI DI NAPOLI FEDERICO II

SSM Scuola Superiore Meridionale

GENERALIZED LOTKA-VOLTERRA EQUATIONS WITH STOCHASTIC DISORDER

S. Suweis, Francesco Ferraro, S. Azaele, A. Maritan

arXiv:2307.02851

University of Padova

francesco.ferraro.4@phd.unipd.it

Generalized Lotka-Volterra equations

$$\frac{dN_i}{dt} = \frac{r_i}{K_i} N_i \left(K_i - N_i - \sum_{j \neq i} \alpha_{ij} N_j \right)$$

Describe well-mixed ecological community with S species

- N_i : number of individuals of species $i = 1, \ldots, S$
- r_i : intrinsic growth rate
- K_i : carrying capacity
- α_{ij} : effect of species j on growth of species i

GLV equations and statistical physics

$$\frac{dN_i}{dt} = N_i \left(1 - N_i - \sum_{j \neq i} \alpha_{ij} N_j \right)$$

- With many species tools from statistical physics can be used
- Usual assumption: interactions α_{ij} are **random** and **fixed in time**

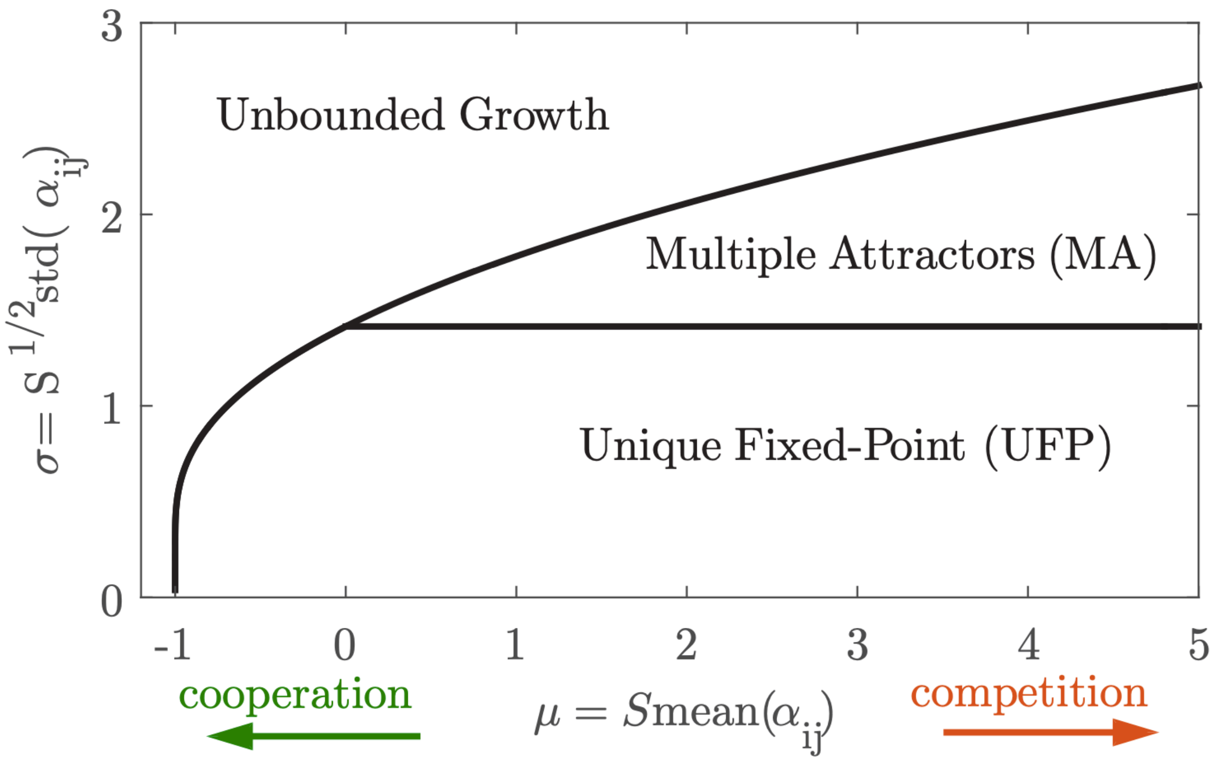
$$\langle \alpha_{ij} \rangle = \frac{\mu}{S} \qquad \text{var}(\alpha_{ij}) = \frac{\sigma^2}{S}$$

Phase diagram of GLV equations

$$\frac{dN_i}{dt} = N_i \left(1 - N_i - \sum_{j \neq i} \alpha_{ij} N_j \right)$$

$$\langle \alpha_{ij} \rangle = \frac{\mu}{S}$$

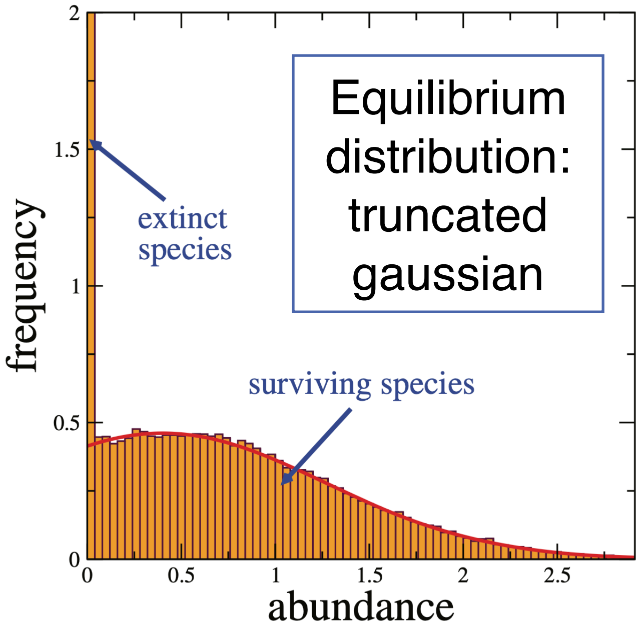
$$\text{var}(\alpha_{ij}) = \frac{\sigma^2}{S}$$

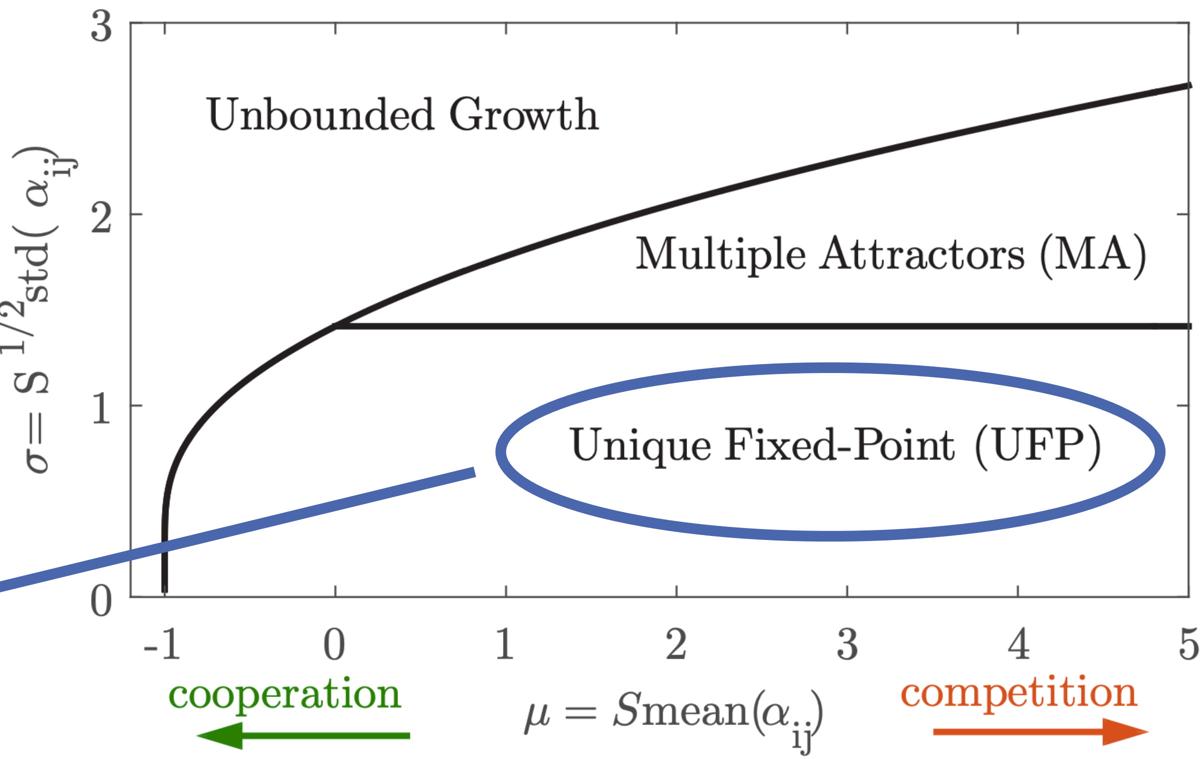


[G. Bunin, PRE 2017]

Equilibrium distribution of GLV equations

$$\frac{dN_i}{dt} = N_i\left(1 - N_i - \sum_{j\neq i} \alpha_{ij} N_j\right)$$



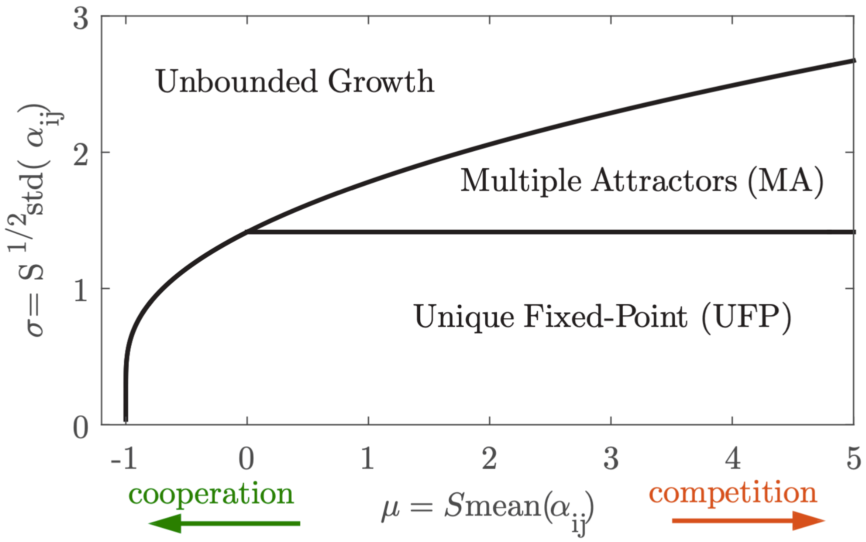


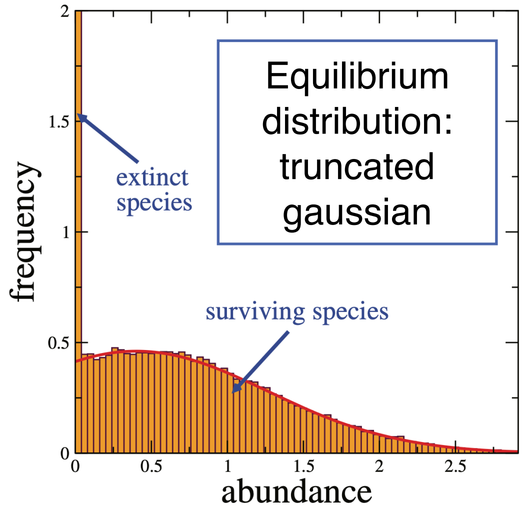
[G. Bunin, PRE 2017, T. Galla, EPL 2018]

Limitations of GLV equations

$$\frac{dN_i}{dt} = N_i\left(1 - N_i - \sum_{j \neq i} \alpha_{ij} N_j\right)$$

- Equilibrium distribution is not realistic
- Unbounded growth is non-physical
- Interactions may change in time





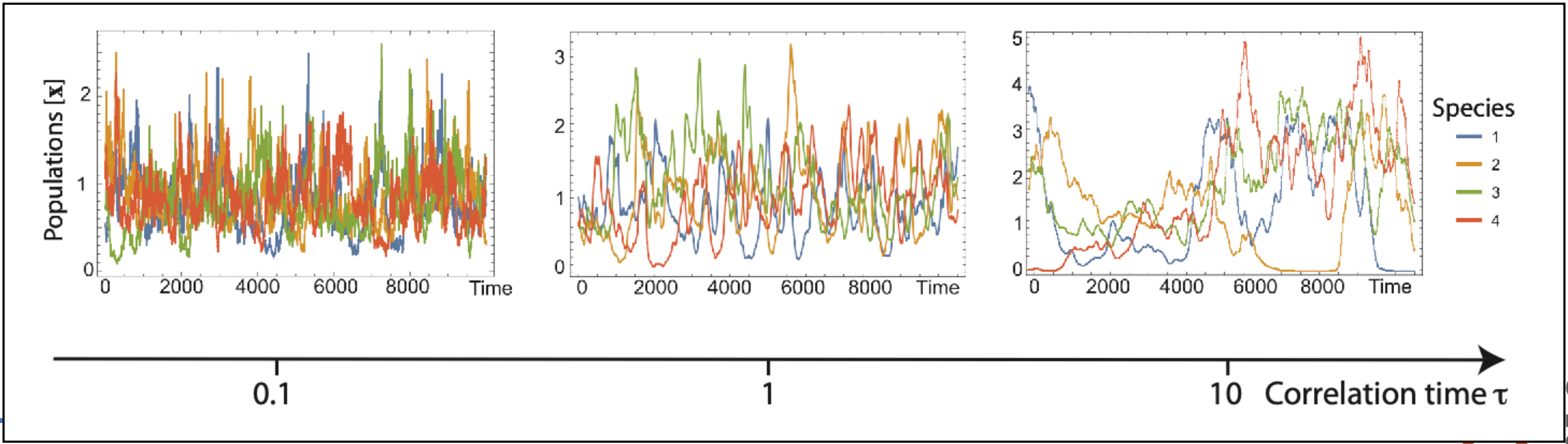
GLV equations with stochastic interactions

$$\frac{dN_i}{dt} = \frac{r_i}{K_i} N_i \left(K_i - N_i - \sum_{j \neq i} \alpha_{ij}(t) N_j \right)$$

$$\alpha_{ij}(t) = \frac{\mu}{S} + \frac{\sigma}{\sqrt{S}} z_{ij}(t)$$

$$\langle z_{ij}(t) \rangle = 0$$

$$\langle z_{ij}(t) z_{ij}(t') \rangle \propto \exp\left(- \frac{t - t'}{\tau} \right)$$



DMFT for GLV with stochastic interactions

- In the limit of large number of species dynamics of community is equivalent to Dynamical Mean-Field Theory

$$\frac{dN_i}{dt} = N_i \left(1 - N_i - \sum_{j \neq i} \alpha_{ij}(t) N_j \right) \quad \Rightarrow \quad \frac{dN}{dt} = N\left[1 - N + \eta(t)\right]$$

$\eta(t)$ Gaussian noise with

$$\langle \eta(t) \rangle = \mu \langle N(t) \rangle$$

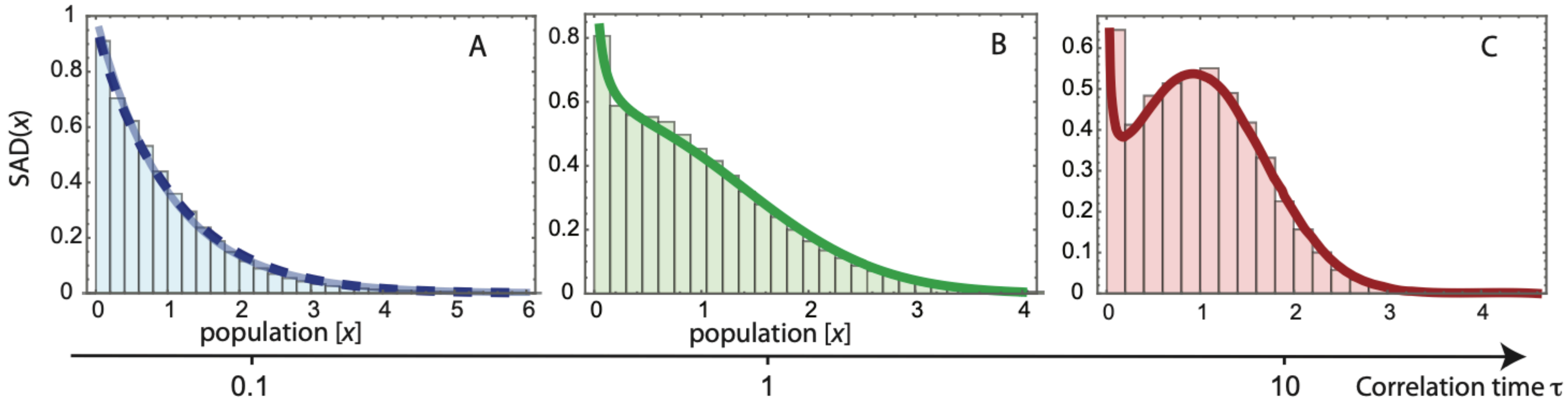
$$\langle \eta(t)\eta(t') \rangle_c = \sigma^2 P(\Delta t) \langle N(t) N(t') \rangle$$

Equilibrium distribution: UCNA approximation

$$dN/dt = N\left[1 - N + \eta(t)\right]$$

- No solution for equilibrium distribution due to colored noise
- With Unified Colored-Noise Approximation:

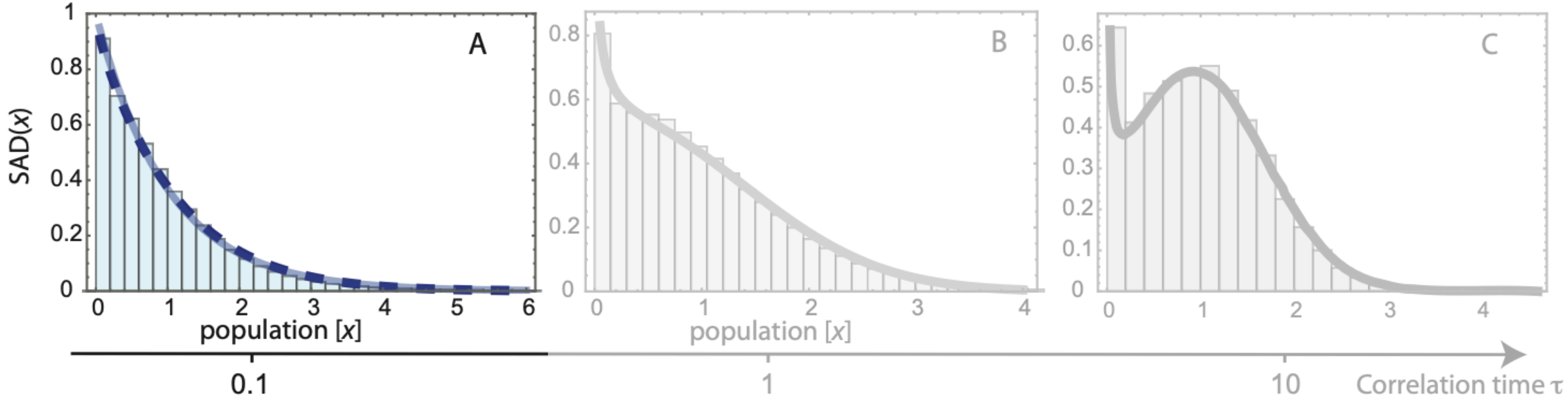
$$P_{eq}(N) \propto \left(\frac{1}{\bar{\tau}} + N\right) N^{-1+\delta} e^{-N/D} \cdot e^{-\bar{\tau}(N-\bar{N})^2/2D}$$



Equilibrium distribution in white-noise limit

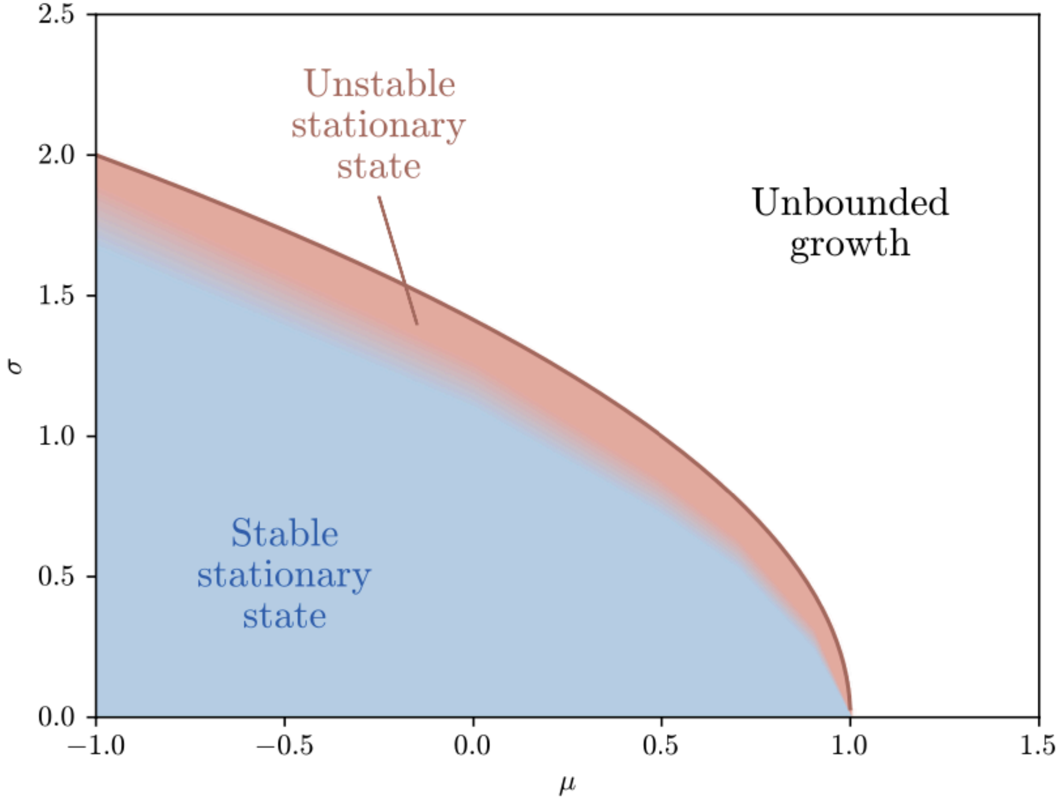
- For $\tau \to 0$ the **exact** equilibrium is

$$P_{eq}(N) \propto N^{-1+\delta} e^{-\beta x}$$



Stability of equilibrium in white-noise limit

- Due to non-linearity of Fokker-Planck equation stationary state is not always reached



Summary of results

- New paradigm of stochastic disorder in GLV equations

- Derivation of DMFT equation
- Equilibrium distribution within UCNA approximation
- Exact equilibrium distribution in the limit $\tau \to 0$
- Numerical insight on stability of white-noise equilibrium

Further work and perspectives

- Why can't the stationary be always reached?
- Unbounded growth still present
- Correlations between different α_{ij}
- Both quenched and stochastic disorder (noisy interactions)
- Application to other equations

Further work and perspectives

- Why can't the stationary be always reached?
- Unbounded growth still present
- Correlations between different α_{ij}
- Both quenched and stochastic disorder (noisy interactions)
- Application to other equations

THANK YOU FOR YOUR ATTENTION!

S. Suweis, F. Ferraro, S. Azaele, A. Maritan

University of Padova

francesco.ferraro.4@phd.unipd.it