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Dynamical behaviour of Brownian particles coupled to a critical Gaussian field

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Abstract

Dynamical properties of a Brownian particle linearly coupled to the local order parameter of a fluid undergoing a continuous phase transition are studied. For a non-conserved order parameter characteristic time scales of various dynamical quantities of the particle are shown to be divergent at criticality. In the case of a globally conserved order parameter these are found to be divergent also away from criticality. Exponents of the resulting long-time algebraic decays are calculated in arbitrary spatial dimension and shown to be universal, using a perturbative approach valid in the weak-coupling limit between the particle and the fluid. Results are illustrated using numerical simulations of a one-dimensional lattice polymer.

Organization of the thesis

In Chapter 1 background and motivations for the work contained in the thesis are given. The model system studied is introduced and justified.

In Chapter 2 we present general properties of this system. The approximation scheme and the simulation algorithm employed in the thesis are also described. Findings are mostly known.

In Chapter 3 we study the dynamical behaviour of a colloid in a harmonic trap. We present novel results about the autocorrelation of the particle position and its relaxation to equilibrium.

In Chapter 4 we give preliminar novel results about the effect of the medium on the correlations between two particles.

In Chapter 5 we conclude with a summary of the results of the thesis and point to directions for future work.

Notation

Vector quantities are not explicitly denoted by bold lettering or arrowing. The Einstein summation convention is sometimes in place. The normalization of the Fourier transform is $f_q = \int \mathrm{d}^d x \, e^{-iqx} f(x)$ and $f(x) = (2\pi)^{-d} \int \mathrm{d}^d q \, e^{iqx} f_q$.

Contents

Chapter 1

Introduction

Many important phenomena in nature appear to be unpredictable or random, thus fluctuations emerge in the description of all sort of physical systems. In statistical physics and, in general, in effective descriptions of physical systems, these fluctuations result from the infeasibility of treating exactly the dynamics of a large number of degrees of freedom, which is often theoretically, experimentally and numerically challenging at best, and moreover usually not really desirable [1, 2]. But even in fundamental physics, fluctuations are intrinsically present due to the uncertainty principle of quantum mechanics [3], which is crucial to explain phenomena on virtually all length and time scales, from the properties of the vacuum [4], to forces between atoms [5], and even up to the radiation emitted by black holes [6].

Macroscopically, fluctuations are suppressed upon increasing the number of constituents of the system, in accordance to the central limit theorem. On the other hand, this means that fluctuations play a more prominent role in mescoscopic systems.

An important example of this kind of systems is a mesoscopic particle suspended in a fluid, moving irregularly as a result of the random collision with the molecules of the solvent. These particles, called *Brownian particles* or *colloids*, have radii ranging from 1 nanometer to 10 microns [7], and are small enough to exhibit thermal motion, but sufficiently large so that they interact with the solvent practically only in an averaged way [8].

This erratic motion was first qualitatively observed by the botanist Robert Brown in pollen grains suspended in water [9], explained theoretically by Einstein [10], Sutherland [11], Smoluchowski [12] and Langevin [13] and quantitatively measured experimentally by Perrin [14]. These seemingly modest works immediately reached landmark status by ending the skepticism around the atomic hypothesis of matter, argued by some to be the single most important fact of modern science [15]. Moreover, after these seminal works, the newly born field of stochastic calculus rapidly became a valuable tool in other areas of physics [16, 17], applied mathematics [18], biology [19], finance [20] and many other fields [21].

As the motion of a colloidal particle is driven by the surrounding fluid, the properties of the first are intrinsically linked to the those of the latter. A class of solvents in which the theory of Brownian motion is applicable in its simplest form is that of *Newtonian* or *purely viscous* fluids. These are characterized by a constant viscosity, independent of the applied stress and the flow velocity [22]. Despite being the simplest model accounting for viscosity, many common liquid and gases, such as water and air, can be assumed to be Newtonian under ordinary conditions. Experimentally, relaxation times in these fluids are known to be of the order of $10^{-14}\,$s [8]. Since the timescales on which a mesoscopic particle evolves are considerably larger, of the order of $10^{-9}\,$s, as consequence of the very large mass of the Brownian particle relatively to that of a solvent molecule, the real movement of a colloid in a Newtonian fluid is remarkably well approximated by the ideal Brownian motion [8].

A colloidal particle suspended in such a fluid is subject to two forces. The first is the total random force $\xi(t)$, or *noise*, resulting from the large number of collisions between the colloid and the molecules of the solvent. In a homogeneous medium this force will average to zero and, given the previous discussion on the timescales, will be shortly correlated, i.e.

$$\langle \xi(t) \rangle = 0 \quad \text{and} \quad \langle \xi(t)\xi(t') \rangle \propto \delta(t-t'), \tag{1.1}$$

the δ-function being an idealization arising in the limit of a solvent evolving infinitely rapidly. The second force acting on the particle is the friction $-\gamma v$, proportional to its velocity, due to the systematic collisions of the particle with the solvent molecules when moving through fluid. The *friction coefficient* γ is a phenomenological parameter and, for example, given by Stokes's law $\gamma = 6\pi\eta R$ in the case of a large sphere of radius R, where η is the viscosity of the fluid.

The equation of motion for the particle is then the so-called *underdamped Langevin equation* [8,13]

$$m\ddot{y} = -\gamma\dot{y} + \xi(t). \tag{1.2}$$

We note that this Langevin equation, that we deduced from physically sensible but ultimately heuristic considerations, can be derived in a more general form from first principles, starting from full microscopic Liouville equation of the solvent and the particle, through the Mori-Zwanzig projection procedure [2,7,23].

In this thesis we consider an approximation of the underdamped equation, valid for times $\gg m/\gamma$, where it can be shown [8] that the momentum is in thermal equilibrium with the solvent and the inertia of the particle can be ignored. Assuming then $m\ddot{y} \approx 0$ we get the so-called *overdamped Langevin equation* [8,16]

$$\dot{y} = -\nu\nabla H + \xi(t), \tag{1.3}$$

where we introduced the *mobility* of the particle $\nu \equiv 1/\gamma$ [24], rescaled the stochastic force $\nu\xi \to \xi$ and added an external potential H. If the solvent is at temperature T, the particle at long times thermalizes in the external potential, reaching the canonical equilibrium distribution

$$P_{\mathrm{eq}}(y) \propto e^{-H(y)/T}. \tag{1.4}$$

This behaviour is reproduced in both the underdamped and overdamped equation if the noise satisfies [25]

$$\langle \xi_i(t)\xi_j(t') \rangle = 2\nu T \delta_{ij} \delta(t - t'), \tag{1.5}$$

where we indicated explicitly the cartesian components $i = 1, \ldots, d$. The fact that the amplitude of the noise is related to the timescale $1/\nu$ on which the relaxation to equilibrium takes place is a well-known instance of the fluctuation-dissipation theorem.

After the pioneering work on Brownian motion, in recent years experimental studies of the motion of colloidal particles in Newtonian fluids have played a significant role in serving as a test bench for various extension of classical thermodynamics to microscopic and non-equilibrium systems [7, 26–33]. The rich insight gained in the study of the motion of a particle in these simple fluids naturally raised the question of how dynamical properties are modified in the presence of a more complex type of solvent. As a matter of fact, a large variety of systems show *non-Newtonian* or *viscoelastic* behaviour, including biological fluids, polymer solutions and micellar systems [34]. These systems present very long relaxation times, that can become comparable to the timescale of Brownian motion [7, 35–38].

The interest in the dynamical properties of mesoscopic particles in these type of fluid is far more than merely theoretical. For example, in industrial settings, *rheological measurements* of deformations and flow of materials are traditionally performed by shearing a macroscopic volume of a sample between two solid surfaces with given geometry [39]. In the last two decades, however, given the necessity to perform measurement of materials, typically of biological origin, difficult to procure in the quantities needed for these measurements, a variety of techniques, known under the name *microrheology* have been developed and investigated [40, 41]. They consist in placing colloids of various nature in a fluid and using them as tracers to probe its properties. The most straightforward technique is to observe how the fluctuations and the diffusion of the particle change with respect to simple Brownian motion [42]. Another approach consist in trapping the particle, usually optically, and observe its relaxational behaviour [43]. A more sophisticated "two-point" technique consists in measuring the correlations between two or more particles [44]. In recent years, accordingly, there has been a great interest in delucidating how the statics and dynamics of such tracer colloidal particle are influenced by complex fluids [45].

Among these are fluids near a second-order critical point. Characterized by fluctuations with large correlation lengths and long relaxation times, they give rise to a plethora of intriguing phenomena [5, 46, 47]. While static properties are studied to some extent, and preliminar result for the dynamic properties are known [48–51], surprisingly the dynamical behaviour of a tracer particle near the critical point of a fluid is an issue which has rarely been addressed in the literature.

In this thesis this problem is tackled, by showing how a variety of dynamical properties of a Brownian particle are modified by the coupling to a fluid undergoing a continuous phase transition.

A *continuous* or *second-order* phase transition is a transition between two different phases of a system characterized statically by a diverging correlation length [52]

$$\xi \propto |T - T_c|^{-\nu} \tag{1.6}$$

where ν is the so-called *correlation length exponent*. In these transitions an *order parameter* ϕ, a measure of the degree of order across the different phases, varies continuously and its spatial correlations decay algebrically at criticality.

In this thesis we will only consider the simplest case of a scalar order parameter. Examples of such order parameters include the coarse-grained magnetization $\phi(x)$ of an Ising-like ferromagnet, the local deviation from the critical density $\phi(x) \propto n(x) - n_c$ in a single component fluid, or the local concentration deviation of one of two species in a binary mixture.

From a dynamical point of view, one also observes a divergence of the characteristic time scale τ on which the order parameter evolves

$$\tau \propto |T - T_c|^{-z\nu}. \tag{1.7}$$

which defines the *dynamic critical exponent* z. This phenomenon, known as *critical slowing-down*, provides us with a natural separation of time scales, as all other physical quantities will fluctuate much faster than ϕ. Effectively, then, we can consider all microscopic dynamical variables as merely constituting a stochastic forcing on the mesoscopic variables, that in our case are the particle position and the order parameter.

A spatially varying scalar order parameter near a critical point will follow an infinite-dimensional analogous of the overdamped Langevin equation (1.3), i.e.

$$\partial_t \phi(x,t) = -D \frac{\delta H[\phi]}{\delta \phi(x,t)} + \zeta(x,t), \tag{1.8}$$

where D denotes the relaxation rate of the field. This equation describes the simplest purely relaxational behaviour of an order parameter and is referred to as *model A* dynamics [53] in the classification of Hohenberg and Halperin [54]. The noise is taken to have zero mean and correlations

$$\langle \zeta(x,t) \zeta(x',t') \rangle = 2DT\delta(x-x')\delta(t-t'), \tag{1.9}$$

which ensure the correct equilibrium distribution. This dynamics is pertinent in describing for example the transition in a single-component fluid.

Alternatively, we may consider also the case in which the order parameter is also globally conserved,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \mathrm{d}x \, \phi(x,t) = 0 \qquad \Longleftrightarrow \qquad \partial_t \phi(q=0,t) = 0, \tag{1.10}$$

which is the case for example in a transition in a binary mixture. The simplest dynamics that is compatible with this conservation law is the so-called *model B* dynamics [53, 54]

$$\partial_t \phi(x,t) = D\nabla^2 \frac{\delta H[\phi]}{\delta \phi(x,t)} + \zeta(x,t), \tag{1.11}$$

and the correlations of the noise are in this case modified to

$$\langle \zeta(x,t)\zeta(x',t')\rangle = -2DT\nabla^2 \delta(x-x')\delta(t-t'). \tag{1.12}$$

So far we have not specified the energy H of the system.

In the following, we may consider the particle trapped in a harmonic potential

$$H_y(y) = \frac{1}{2}ky^2 \tag{1.13}$$

which can be experimentally realized through optical traps [55].

As the effective potential representing the restoring force for the order parameter we choose a Landau-Ginzburg hamiltonian [53]

$$H_\phi[\phi] = \int \mathrm{d}^d x \left[\frac{1}{2}[\nabla\phi(x)]^2 + \frac{r}{2}\phi(x)^2 + \frac{u}{4!}\phi(x)^4 + \dots\right] \tag{1.14}$$

which is the most general local functional compatible with the symmetries of the system. The higher omitted powers in Eq. (1.14) are irrelevant in a renormalization group sense [53], while the non-linearities represented by the quartic term are not, especially upon approaching the critical point $r = 0$. However, given the preliminarity of our work, in this thesis we truncate the Hamiltonian at the simpler second order, therefore considering a Gaussian field $H_\phi = \frac{1}{2}\phi\Delta\phi$, with $\Delta \equiv -\nabla^2 + r$.

In order to represent the interaction between the particle and the field we make the simplest choice, which is a linear coupling

$$H_{int}[\phi, y] = -\lambda \int \mathrm{d}^d x\, \phi(x) V(x-y). \tag{1.15}$$

This type of interaction corresponds to a local bulk external field acting on the order parameter, which is to be interpreted as a chemical potential in the case of a single-component fluid [56, 57] or a preferential absorption of one of the components on the colloid in the case of a binary mixture [57]. If the scalar field represents the coarse-grained magnetization in a Ising-like ferromagnetic transition, we can think of the particle as a mesoscopic impurity inducing a local magnetic field on the material.

With all this in mind then the total energy of the system is then given by

$$H = H_\phi + H_y + H_{int} \tag{1.16}$$

Upon inserting Eqs. (1.13-1.15) into the ones for the field and the particle Eqs. (1.3, 1.8, 1.11) we get the equations of motion of our system that will be studied in the rest of the thesis:

$$\partial_t \phi(x,t) = -A\phi(x,t) + \lambda D(i\nabla)^\alpha V(x-y(t)) + \zeta(x,t) \tag{1.17}$$
$$\dot{y}(t) = -\nu k y(t) + \lambda\nu f(t) + \xi(t) \tag{1.18}$$

where we defined

$$A \equiv D(i\nabla)^\alpha (r - \nabla^2) \tag{1.19}$$
$$f(t) \equiv \nabla_y \int \mathrm{d}^d x\, \phi(x,t) V(x-y(t)) \tag{1.20}$$

We also introduced a convenient notation for field the includes both model A dynamics, for $\alpha = 0$, and model B dynamics, for $\alpha = 2$.

Chapter 2

General properties of the system

In this Chapter we detail some properties of the dynamical system defined by Eqs. (1.17) and (1.18). We start by deriving well-known features of the uncoupled i.e., $\lambda = 0$, particle and field dynamics. We then show non-perturbatively that static properties of the particle are not affected by the interaction with the order parameter. The tools of our investigation for the rest of the thesis, a weak-coupling perturbative expansion and a simulation algorithm, are also introduced. The first is then employed to derive the effective diffusion constant of the coupled Brownian motion, while the latter is used to confirm numerically the analytical result on the equilibrium distribution of the colloid.

2.1 Uncoupled free particle: Wiener process

Consider the equation of motion Eq. (1.18) for the particle when it is not coupled to the scalar field i.e., $\lambda = 0$ and free i.e., $k = 0$.

The particle in this case performs a pure Brownian motion and the stochastic process describing its position in time is known as the *Wiener process*. As discussed in the introduction, this process is appropriate for the description of a colloidal particle immersed in a Newtonian fluid with very short relaxation times, and can be equivalently obtained as the scaling limit of an unbiased random walk [53] or by Wick rotating a free quantum particle into its Euclidean formulation [58].

In this situation Eq. (1.18) simply reduces to

$$\dot{y}_i = \xi_i(t), \tag{2.1}$$

where the statistical properties of the Gaussian noise are fixed by its first two moments

$$\begin{aligned} \langle \xi_i(t) \rangle &= 0, \\ \langle \xi_i(t) \xi_j(t') \rangle &= 2\nu T \delta_{ij} \delta(t - t'). \end{aligned} \tag{2.2}$$

The quantity ν is the mobility of the particle, and we recall that its appearance in the amplitude of the noise is a consequence of the fluctuation-dissipation theorem.

The noise can be equivalently defined by the functional probability distribution

$$P[\xi] \propto \exp\left[-\frac{1}{4\nu T}\int \mathrm{d}t\, \xi^2(t)\right]. \tag{2.3}$$

We can formally solve Eq. (2.1) as

$$y_i(t) = y_{0,i} + \int_{t_0}^{t} \mathrm{d}s\, \xi_i(s), \tag{2.4}$$

where t_0 is the time at which the initial condition $y_{0,i}$ is imposed. With this the first two moments of the process are easily computed

$$\begin{aligned} \langle y_i(t)\rangle &= y_0, \\ \langle y_i(t) y_j(t')\rangle &= 2\nu T[\min(t,t') - t_0]\, \delta_{ij}. \end{aligned} \tag{2.5}$$

Since the process $y_i(t)$ is a linear combination of the Gaussian process $\xi_i(t)$ at different times, it is itself Gaussian [59], a fact which has a number of consequences. The mean and the autocorrelation of the process specify it completely, and higher moments can be determined via Wick's theorem [16, 53]. The fact that the Cartesian components of the position of the particle are uncorrelated implies that they are also independent. Moreover, the probability distribution of the position $y(t)$ of the particle at time t given its position y_0 at time t_0, known as the *propagator* $P_{1|1}(y,t|y_0,t_0)$ [53], is Gaussian. Given that

$$\langle \left[y_i(t) - \langle y_i(t)\rangle\right]^2 \rangle = 2\nu T(t - t_0), \tag{2.6}$$

the propagator of the Wiener process is then

$$P_{1|1}(y,t|y_0,t_0) = \frac{1}{\sqrt{4\pi\nu T(t-t_0)}} \exp\left[-\frac{(y-y_0)^2}{4\nu T(t-t_0)}\right]. \tag{2.7}$$

We draw attention to Eq. (2.6), which expresses the distinguishing property of Brownian motion of a mean square displacement growing linearly in time.

We also notice that the propagator of the Wiener process with our normalization satisfies the *Fokker-Planck equation* [60], with $\rho(x,t) = P_{1|1}(x,t|x_0,t_0)$,

$$\begin{cases} \partial_t \rho = \nu T \nabla^2 \rho, \\ \rho(x,t_0) = \delta(x - x_0). \end{cases} \tag{2.8}$$

2.2 Uncoupled trapped particle: Ornstein-Uhlenbeck process

Consider again the particle not coupled to the field i.e., $\lambda = 0$, but this time harmonically trapped i.e., $k > 0$.

The resulting process is known as the *Ornstein-Uhlenbeck process*. We note that this process describes also the evolution of the momentum of a underdamped Langevin particle, as it follows by comparing Eq. (1.2) and Eq. (2.9). We emphasizes, however, that the two equations are only incidentally the same and describe different physical situations. The Ornstein-Uhlenbeck process can be obtained as the Euclidean continuation of the quantum harmonic oscillator [58].

The equation (1.18) for the evolution of the coordinate of the particle, in this case, is

$$\dot{y}_i(t) = -\nu k y_i(t) + \xi_i(t), \tag{2.9}$$

where the correlations of the noise are the same as before, Eq. (2.2). This equation of motion can be formally solved as

$$y_i(t) = e^{-\nu k(t-t_0)} y_{0,i} + \int_{t_0}^{t} \mathrm{d}s\, e^{-\nu k(t-s)} \xi(s), \tag{2.10}$$

so its the first two moments are computed to be

$$\langle y_i(t) \rangle = e^{-\nu k(t-t_0)} y_0, \tag{2.11}$$

$$\langle y_i(t) y_j(t') \rangle = \frac{T}{k} e^{-\nu k|t-t'|}\, \delta_{ij}. \tag{2.12}$$

For the same reasons as for the Wiener process, this process is also Gaussian, and also in this case its components are independent. From

$$\langle \big[y_i(t) - \langle y_i(t) \rangle \big]^2 \rangle = \frac{T}{k} \big[1 - e^{-2\nu k(t-t_0)} \big], \tag{2.13}$$

the propagator can be computed explicitly

$$P_{1|1}(y, t|y_0, t_0) = \frac{1}{\sqrt{2\pi T/k \left(1 - e^{-2\nu k(t-t_0)}\right)}} \exp\left[-\frac{\left(y - y_0\, e^{-\nu k(t-t_0)}\right)^2}{2T/k \left(1 - e^{-2\nu k(t-t_0)}\right)} \right]. \tag{2.14}$$

The equilibrium distribution of the particle can be found by letting $t_0 \to -\infty$ and is, as anticipated, the canonical one

$$P_{\mathrm{eq}}(y) = \frac{1}{\sqrt{2\pi T/k}} \exp\left[-\frac{k y^2}{2T} \right]. \tag{2.15}$$

The Ornstein-Uhlenbeck process satisfies the Fokker-Planck equation

$$\begin{cases} \partial_t \rho = \nu k \nabla(x\rho) + \nu T \nabla^2 \rho, \\ \rho(x, 0) = \delta(x - x_0), \end{cases} \tag{2.16}$$

where $\rho(x, t) = P_{1|1}(x, t|x_0, t_0)$.

For the purposes of the analysis of the following sections and chapters, for a finite t_0, we only need the average

$$
\begin{aligned}
Q_q(t,t') &\equiv \langle e^{iq\cdot(y(t)-y(t'))}\rangle \\
&= \int \mathrm{d}^d y\, \mathrm{d}^d y'\, e^{iq\cdot(y-y')} P_{1|1}(y,t|y_0,t_0) P_{1|1}(y',t'|y,t) \\
&= \exp\left[iq\cdot y_0 \Big(e^{-\nu k(t-t_0)} - e^{-\nu k(t'-t_0)} \Big) - \frac{Tq^2}{k} R(t,t') \right],
\end{aligned}
\tag{2.17}
$$

where we used the explicit expressions of the propagator reported above and where we defined

$$
R(t,t') \equiv 1 - e^{-\nu k|t-t'|} - \frac{1}{2}\Big[e^{-\nu k(t-t_0)} - e^{-\nu k(t'-t_0)} \Big]^2. \tag{2.18}
$$

In the stationary state, corresponding to $t_0 \to -\infty$, we will also need

$$
\begin{aligned}
\langle e^{iq\cdot y(t)} e^{iq'\cdot y(t')}\rangle &= \int \mathrm{d}^d y\, \mathrm{d}^d y'\, e^{iq\cdot y} e^{iq' y'} P_{\mathrm{eq}}(y) P_{1|1}(y't'|yt) \\
&= \exp\left[-\frac{T}{2k} \Big(q^2 + 2q\cdot q' e^{-\nu k|t-t'|} + q'^2 \Big) \right],
\end{aligned}
\tag{2.19}
$$

$$
Q_q^{\mathrm{eq}}(t-t') \equiv \langle e^{iq\cdot(y(t)-y(t'))}\rangle = \exp\left[-\frac{T}{k} \Big(1 - e^{-\nu k|t-t'|} \Big) q^2 \right], \tag{2.20}
$$

$$
\begin{aligned}
\langle e^{iq\cdot(y(t)-y(t'))} y_i(s)\rangle &= \int \mathrm{d}^d y\, \mathrm{d}^d y'\, \mathrm{d}^d y''\, P_{\mathrm{eq}}(y) P_{1|1}(y',t'|y,t) P_{1|1}(y'',s|y',t') \\
&= \frac{T}{k}\left[e^{-\nu k|s-t|} - e^{-\nu k|s-t'|} \right] iq_i Q_q^{\mathrm{eq}}(t-t').
\end{aligned}
\tag{2.21}
$$

We emphasize that Eqs. (2.17)–(2.21) are valid in an arbitrary number of dimensions.

2.3 Uncoupled field

The equation of motion Eq. (1.17) for the order parameter when is not coupled to the particle i.e., for $\lambda = 0$, reduces to

$$
\partial_t \phi(x,t) = -D(i\nabla)^\alpha (r - \nabla^2)\phi(x,t) + \zeta(x,t). \tag{2.22}
$$

We recall that $\alpha = 0$ renders model A dynamics while $\alpha = 2$ corresponds to model B dynamics.

This equation can be formally solved as

$$
\phi(x,t) = \int_{-\infty}^{t} \mathrm{d}s\, e^{-A(t-s)} \zeta(x,t) \quad \Longleftrightarrow \quad \phi_q(t) = \int_{-\infty}^{t} \mathrm{d}s\, e^{-A_q(t-s)} \zeta_q(t), \tag{2.23}
$$

where for brevity we introduced the differential operator A and its Fourier transform

$$A = D(i\nabla)^\alpha (r - \nabla^2) \qquad \Longleftrightarrow \qquad A_q = Dq^\alpha (r + q^2). \tag{2.24}$$

The expression of this operator in Fourier space demonstrates manifestly that it is positive-definite, since $A_q \geqslant 0$.

Notice that here, as in the rest of the thesis, we take the initial condition of the field in the infinite past or, equivalently, we consider a field which is initially equilibrated. This renders the subsequent calculations less cumbersome and physically corresponds to inserting the colloid in a medium which has been previously equilibrated.

With the explicit solution for $\phi_q(t)$ it is straightforward to compute the Fourier-space correlation function

$$\langle \phi_q(t)\phi_{q'}(t')\rangle = (2\pi)^d \delta(q + q')\, G_q(t - t'), \tag{2.25}$$

where we introduced the Fourier-space propagator

$$G_q(t) = \frac{T}{r + q^2} e^{-A_q|t|}. \tag{2.26}$$

The fact that the correlation function in Eq. (2.25) contains a δ-function is a simple consequence of the translational invariance of the system, encoded in the real space propagator

$$G(x, t) = \langle \phi(x, t)\phi(0, 0)\rangle. \tag{2.27}$$

From Eq. (2.26) we see that each Fourier mode contributes to the correlation function of the scalar field in real space with a term which decays on a a characteristic scale

$$\tau = \begin{cases} 1/[D(r + q^2)] & \text{for model A,} \\ 1/[Dq^2(r + q^2)] & \text{for model B,} \end{cases} \tag{2.28}$$

This time scale increases upon approaching the critical point $r = 0$, reflecting the phenomenon of critical slowing-down [53], especially severe as $q \to 0$, i.e., for the fluctuation modes with the longest wavelength. We note that away from criticality this characteristic time scale remains finite for all modes in the case of model A, while it diverges for $q \to 0$ in model B, as a consequence of the global conservation.

Exactly at the critical point $r = 0$ this characteristic time scale is considearbly enhanced

$$\tau = \begin{cases} 1/(Dq^2) & \text{for model A,} \\ 1/(Dq^4) & \text{for model B,} \end{cases} \tag{2.29}$$

which, importantly, diverges for $q \to 0$ for both model A and model B.

2.4 Equilibrium distribution of coupled particle

The equilibrium distribution of the system constituted by the colloidal particle in interaction with the field is given by the Boltzmann distribution

$$P[\phi, y] \propto \exp(-H[\phi, y]/T). \tag{2.30}$$

The corresponding equilibrium distribution of the colloid can be found by simply marginalizing

$$P(y) = \frac{\int [\mathrm{d}\phi] e^{-H[\phi,y]/T}}{\int \mathrm{d}y\, [\mathrm{d}\phi] e^{-H[\phi,y]/T}} = e^{-H_y(y)} \frac{\int [\mathrm{d}\phi] e^{-H_\phi[\phi,y]/T}}{\int \mathrm{d}y\, [\mathrm{d}\phi] e^{-H[\phi,y]/T}}. \tag{2.31}$$

If the field is Gaussian, as assumed here, the integration over the field configurations can be performed explicitly

$$\begin{aligned} \int [\mathrm{d}\phi] e^{-H_\phi[\phi,y]/T} &= \int [\mathrm{d}\phi] \exp\left[-\frac{1}{T}\int \mathrm{d}x \left(\frac{1}{2}\phi(x)\Delta\phi(x) - \lambda\phi(x)V(x-y)\right)\right] \\ &\propto \exp\left[\frac{\lambda^2}{2T}\int \mathrm{d}x\,\mathrm{d}x'\, V(x-y)\Delta^{-1}(x-x')V(x'-y)\right] \\ &= \exp\left[\frac{\lambda^2}{2T}\int \mathrm{d}x\,\mathrm{d}x'\, V(x)\Delta^{-1}(x-x')V(x')\right], \end{aligned} \tag{2.32}$$

where in the last equality we simply shifted $x \to x - y$, $x' \to x' - y$.

This calculation shows explicitly that, even in presence of an interaction between the field and the particle the equilibrium distribution of the latter is unaltered compared to the case in which that interaction is absent, so that

$$P_{\mathrm{eq}}(y) \propto e^{-H_y(y)/T} \quad \text{for any } \lambda. \tag{2.33}$$

In particular, this also implies that any property of the field-particle interaction can only be deduced from dynamical properties of the probe.

We anticipate that this result for the equilibrium distribution is numerically verified remarkably well, see Section 2.7.3

2.5 Weak-coupling approximation

The coupled nonlinear equations Eqs. (1.17) and (1.18) for the particle and the field are not solvable exactly in general and we therefore resort to a perturbative expansion of the equations of motion in the coupling strength λ, and compute the relevant observables at the lowest order in this parameter.

In practice we consider the following formal expansions for the field and the coordinates of the particle:

$$\begin{aligned} \phi(x,t) &= \sum_{n\geqslant 0} \lambda^n \phi^{(n)}(x,t), \\ y(t) &= \sum_{n\geqslant 0} \lambda^n y^{(n)}(t). \end{aligned} \tag{2.34}$$

By inserting these expansions in the equations of motion Eq. 1.18 we get for the particle

$$\begin{aligned} \dot{y}^{(0)}(t) &= -\nu k y^{(0)}(t) + \xi(t), \\ \dot{y}^{(1)}(t) &= -\nu k y^{(1)}(t) + \nu f^{(0)}(t), \\ \dot{y}^{(2)}(t) &= -\nu k y^{(2)}(t) + \nu f^{(1)}(t), \end{aligned} \tag{2.35}$$

where we defined

$$f_i^{(0)}(t) \equiv \nabla_{y_i} \int \mathrm{d}^d x\, \phi^{(0)}(x,t) V(x - y^{(0)}(t)), \tag{2.36}$$

$$f_i^{(1)}(t) \equiv \nabla_{y_i} \int \mathrm{d}^d x \left[\phi^{(1)}(x,t) V(x - y^{(0)}(t)) - \phi^{(0)}(x,t) \nabla_j V(x - y^{(0)}(t)) y_j^{(1)}(t) \right]. \tag{2.37}$$

In these equations we are employing the Einstein summation convention on the j indices.

At zeroth order the equation for the praticle is solved by the Wiener process or an Ornstein-Uhlehnbeck process, as discussed in the previous sections.

The higher-order equations for the coordinate of particle can be formally solved as

$$y^{(1)}(t) = \nu \int_{t_0}^{t} \mathrm{d}s\, e^{-\nu k(t-s)} f^{(0)}(s), \tag{2.38}$$

$$y^{(2)}(t) = \nu \int_{t_0}^{t} \mathrm{d}s\, e^{-\nu k(t-s)} f^{(1)}(s). \tag{2.39}$$

Similarly, the equations of motion for the field derived from Eq. (1.17) are

$$\partial_t \phi^{(0)}(x,t) = -A\phi^{(0)}(x,t) + \zeta(x,t), \tag{2.40}$$

$$\partial_t \phi^{(1)}(x,t) = -A\phi^{(1)}(x,t) + D(i\nabla)^\alpha V(x - y^{(0)}(t)). \tag{2.41}$$

The properties at lowest order of the uncoupled field have already been discussed.

The first-order equation of motion of the field can be formally solved as

$$\phi^{(1)}(x,t) = D \int_{-\infty}^{t} \mathrm{d}s\, e^{-A(t-s)} (i\nabla)^\alpha V(x - y^{(0)}(s)) \tag{2.42}$$

2.6 Calculation of the diffusion constant

As an application of the weak-coupling approximation scheme introduced above, we calculate the diffusion constant of the colloid at the lowest perturbative order in the coupling strength λ. Our result is shown to coincide with results known in the literature, obtained though through other means [48, 61].

What we aim to compute is

$$\begin{aligned}\langle y(t)^2\rangle &= \langle y^{(0)}(t)^2\rangle + \lambda^2\left[\langle y^{(1)}(t)^2\rangle + 2\langle y^{(2)}(t)y^{(0)}(t)\rangle\right] \\ &= \sum_i \left\{\langle y_i^{(0)}(t)^2\rangle + \lambda^2\left[\langle y_i^{(1)}(t)^2\rangle + 2\langle y_i^{(2)}(t)y_i^{(0)}(t)\rangle\right]\right\}\end{aligned} \tag{2.43}$$

The lowest order has already been computed in Eq. (2.13)

$$\langle y(t)^2\rangle = 2d\nu Tt \equiv dD_0 t. \tag{2.44}$$

where d is the dimensionality of the system.

At the second order, by employing Eq. (2.38) specialized to the case $k = 0$

$$\langle y_i^{(1)}(t)^2\rangle = \nu^2 \int_0^t \mathrm{d}s \int_0^t \mathrm{d}s' \, \langle f_i^{(0)}(s) f_i^{(0)}(s')\rangle. \tag{2.45}$$

In order to compute averages like this one it is convenient to pass to Fourier space

$$f_i^{(0)}(s) = \nabla_{y_i} \int \mathrm{d}^d x \, \phi^{(0)}(x,s) V(x - y^{(0)}(s)) = \int \frac{\mathrm{d}^d q}{(2\pi)^d} i q_i V_q^* e^{iq\cdot y^{(0)}(s)} \phi_q^{(0)}(s). \tag{2.46}$$

Inserting this expression into Eq. (2.45) and evaluating the averages on the uncoupled particle and field one gets with some straightforward computation

$$\begin{aligned}\langle y^{(1)}(t)^2\rangle &= \nu^2 \int_0^t \mathrm{d}s \int_0^t \mathrm{d}s' \int \frac{\mathrm{d}^d q}{(2\pi)^d} q^2 |V_q|^2 e^{-\nu T q^2 |s-s'|} G_q(s-s') \\ &= 2\nu^2 C_d \int_0^t \mathrm{d}s \int_0^s \mathrm{d}s' \int_0^\infty \mathrm{d}q \, q^{d+1} |V_q|^2 e^{-\nu T q^2 (s-s')} G_q(s-s'),\end{aligned} \tag{2.47}$$

where the integration over the angular variables Ω_d is

$$C_d \equiv \int \frac{\mathrm{d}\Omega_d}{(2\pi)^d} = \begin{cases} 1/\pi & d=1 \\ 1/2\pi & d=2 \\ 1/2\pi^2 & d=3. \end{cases} \tag{2.48}$$

For the other term second order term of Eq. (2.43) one proceeds similarly and finds

$$\langle y_i^{(2)}(t) y_i^{(0)}(t)\rangle = \nu \int_0^t \mathrm{d}s \, \langle f_i^{(1)}(s) y_i^{(0)}(t)\rangle. \tag{2.49}$$

The calculation of this last average is slightly more involved than the previous one but is be done in a similar manner. The final result is

$$\begin{aligned}\langle y^{(2)}(t)\cdot y^{(0)}(t)\rangle &= -\int_0^t \mathrm{d}s\int_0^s \mathrm{d}s'\int \frac{\mathrm{d}^d q}{(2\pi)^d} q^2|V_q|^2 2\nu^2(s-s')e^{-\nu Tq^2(s-s')}\\ &\qquad\qquad\qquad\qquad\cdot G_q(s-s')(A_q+\nu Tq^2)\\ &= -2\nu^2 C_d\int_0^t \mathrm{d}s\int_0^s \mathrm{d}s'\int_0^\infty \mathrm{d}q\, q^{d+1}|V_q|^2(s-s')e^{-\nu Tq^2(s-s')}\\ &\qquad\qquad\qquad\qquad\cdot G_q(s-s')(A_q+\nu Tq^2),\end{aligned} \tag{2.50}$$

where C_d was defined in Eq. (2.48)

2.6.1 Asymptotic behaviour at long times

The asymptotic at long times of Eqs. (2.47) and (2.50) can be found from the following estimation, valid for $a > 0$, for $t \to \infty$

$$\begin{aligned}\int_0^t \mathrm{d}s\int_0^s \mathrm{d}s'\, e^{-a(s-s')} &\to a^{-1}t,\\ \int_0^t \mathrm{d}s\int_0^s \mathrm{d}s'\,(s-s')e^{-a(s-s')} &\to a^{-2}t,\end{aligned} \tag{2.51}$$

where the first can be verified by simply performing the integration and the second follows from taking the derivative of the first. With these expressions at hand, it is straightforward to show that

$$\langle y^{(1)}(t)^2\rangle \to D_2 t \quad \text{and} \quad \langle y^{(2)}(t)\cdot y^{(0)}(t)\rangle \to -D_2 t, \tag{2.52}$$

where we defined

$$D_2 \equiv D_0\int_0^\infty \mathrm{d}q\,\frac{\nu C_d q^{d+1}|V_q|^2}{(r+q^2)(A_q+\nu Tq^2)}, \tag{2.53}$$

so that

$$D(\lambda) = D_0 - \lambda^2 D_2 + o(\lambda^4). \tag{2.54}$$

This result is what was found in Ref. [61] via a Kubo formula formalism and in Ref. [48] via a path-integral calculation. For completeness, a critical derivation of the latter is given in Appendix B.

From this result we infer that that the coupling to the field reduces the diffusivity of the particle, since D_2 is always positive. Moreover, it is not difficult to show that, for both model A and model B dynamics, D_2 diverges upon approaching the critical point $r = 0$. This signals that at the fluid criticality the diffusion of the particle stops being $\langle y(t)^2\rangle \sim t$ and becomes anomalous. We refer to the literature for the interesting discussion of the physical basis of these phenomena [48].

2.7 Simulation algorithm and lattice polymer dynamics

In order to test the various theoretical predictions of the thesis, and in particular to assess the applicability of the weak-coupling approximation, we numerically simulate the colloid-field system. For simplicity, we simulate a one-dimensional system only. To do so, the field is discretized on an evenly spaced lattice, and the particle is modeled by a random walker on this lattice.

2.7.1 Simulation of the field dynamics

The equation of motion for a model A field, assuming a point-like interaction between the field and the particle i.e., $V(x-y)=\delta(x-y)$, is

$$\partial_t \phi(x,t) = D(\nabla^2 - r)\phi(x,t) - D\lambda\delta(x-y(t)) + \zeta(x,t). \tag{2.55}$$

The discretization of this equation is performed by taking a simple forward difference for both time and spatial derivatives, which results in

$$\frac{\phi_{x,t+\Delta t}-\phi_{x,t}}{\Delta t} = D\frac{\phi_{x+\Delta x,t}+\phi_{x-\Delta x,t}-2\phi_{x,t}}{\Delta x^2} - Dr\phi_{x,t} - D\lambda\frac{\delta_{x,y_t}}{\Delta x} + \zeta_{x,t}. \tag{2.56}$$

The discrete noise is also taken to be Gaussian, with moments

$$\langle \zeta_{x,t}\rangle = 0 \qquad \text{and} \qquad \langle \zeta_{x,t}\zeta_{x',t'}\rangle = \frac{2DT}{\Delta x \Delta t}\delta_{xx'}\delta_{tt'}. \tag{2.57}$$

In practice, for each point of the lattice and at each timestep, random numbers are independently drawn from a Gaussian distribution with zero mean and variance $2DT/\Delta x\Delta t$.

Interestingly, the evolution equation (2.56) is precisely that of the Rouse model of a lattice polymer [62]. This model, arguably the conceptually simplest but also most important model of polymer dynamics, is widely employed to model chains in which excluded volume effects are not relevant [63].

In this interpretation of Eq. (2.56) the variable ϕ_x is to be interpreted as the (continuous) displacement of the monomer at point x on the lattice from its equilibrium position. Additionally, each monomer is harmonically bounded in a potential of stiffness $2Dr$ and a contact interaction with the particle is present.

In our simulations we employed periodic boundary conditions in space.

2.7.2 Simulation of particle dynamics

The particle is modeled as a random walker on the lattice polymer. To reproduce the correct dynamics, the probabilities for the particle to move on neighbouring lattice sites or to stay still are taken to be

$$\begin{aligned} P(x\to x\pm\Delta x) &= 1-\exp\left[-W(x\to x\pm\Delta x)\Delta t\right] \approx W(x\to x\pm\Delta x)\Delta t,\\ P(x\to x) &= 1-P(x\to x+\Delta x)-P(x\to x-\Delta x), \end{aligned} \tag{2.58}$$

where the transition rates are

$$W(x \to y) = \frac{\nu T}{\Delta x^2} \min \left[1, \exp\left(-\frac{H(y) - H(x)}{T} \right) \right]. \tag{2.59}$$

The most straightforward way to show that with these rates the random walker follows the continuous overdamped dynamics is to compare their generators. We briefly recall that the *backward generator* L of a stochastic process is the operator

$$Lf(x_t) = \lim_{\Delta t \to 0} \frac{\langle f(x_{t+\Delta t}) \rangle_{x_t = x} - f(x)}{\Delta t}. \tag{2.60}$$

Denoting by $\rho(x,t)$ the probability density of the process at time t

$$\begin{aligned} \frac{\mathrm{d}\langle f(x_t) \rangle}{\mathrm{d}t} &= \lim_{\Delta t \to 0} \frac{\langle f(x_{t+\Delta t}) \rangle - \langle f(x_t) \rangle}{\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left(\int \mathrm{d}x \, \langle f(x_{t+\Delta t}) \rangle_{x_t = x} \rho(x,t) - \int \mathrm{d}x \, f(x) \rho(x,t) \right) \\ &= \int \mathrm{d}x \lim_{\Delta t \to 0} \frac{\langle f(x_{t+\Delta t}) \rangle_{x_t = x} - f(x)}{\Delta t} \rho(x,t) \\ &= \langle Lf(x_t) \rangle, \end{aligned} \tag{2.61}$$

which, in turn, implies that

$$\int \mathrm{d}x \, f(x) \partial_t \rho(x,t) = \frac{\mathrm{d}}{\mathrm{d}t} \int \mathrm{d}x \, f(x) \rho(x,t) = \int \mathrm{d}x \, Lf(x) \rho(x,t) = \int \mathrm{d}x \, f(x) L^\dagger \rho(x,t), \tag{2.62}$$

or, since f is arbitrary,

$$\partial_t \rho = L^\dagger \rho. \tag{2.63}$$

This evolution equation completely specifies the process, and in particular two processes with the same generator are equal.

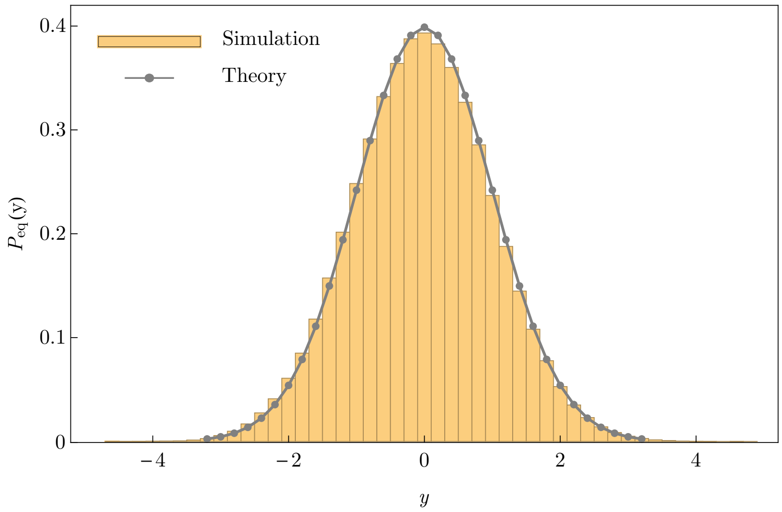

Figure 2.1: Numerical evaluation of the equilibrium distribution of a harmonically trapped colloid, analitycally calculated in Eq. (2.33). The parameters of the simulation are $T, \nu, k, r = 1$, $D = 0.1$ $\lambda = 5$, $N = 100$, $\Delta x = 0.2$, $\Delta t = 0.001$. In this regime of strong coupling and slowly evolving field the dynamical properties of the colloid are expected to depart significantly from the uncoupled case. The histogram is that of a single particle trajectory comprised of 10^8 steps, with data collected after a period of initial equilibration.

The generator of our random walk is

$$\begin{aligned}
Lf(x_t) &= \lim_{\Delta t \to 0} \frac{\langle f(x_{t+\Delta t}) - f(x_t)\rangle_{x_t = x}}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \sum_y p(x \to y)(f(y) - f(x)) \\
&= \sum_y W(x \to y)(f(y) - f(x)) \\
&= W(x \to x + \Delta x)(f(x + \Delta x) - f(x)) \\
&\quad + W(x \to x - \Delta x)(f(x - \Delta x) - f(x)) \\
&= \frac{\nu T}{\Delta x^2} \Bigg[\min\left(1, 1 - \frac{H'(x)}{T}\Delta x\right) \left(f'(x)\Delta x + \frac{1}{2} f''(x)\Delta x^2\right) \\
&\qquad + \min\left(1, 1 + \frac{H'(x)}{T}\Delta x\right) \left(-f'(x)\Delta x + \frac{1}{2} f''(x)\Delta x^2\right) \Bigg] + \dots \\
&= \frac{\nu T}{\Delta x^2} \Bigg[\left(1 - \frac{|H'(x)| + H'(x)}{2T}\Delta x\right) \left(f'(x)\Delta x + \frac{1}{2} f''(x)\Delta x^2\right) \\
&\qquad + \left(1 - \frac{|H'(x)| - H'(x)}{2T}\Delta x\right) \left(-f'(x)\Delta x + \frac{1}{2} f''(x)\Delta x^2\right) \Bigg] + \dots \\
&= -\nu H'(x) f'(x) + \nu T f''(x) + \dots ,
\end{aligned} \tag{2.64}$$

which coincides with the well-known generator of overdamped dynamics in the limit of small Δx. Notice that this remains valid also in the more general case of a time-dependent hamiltonian H.

2.7.3 Numerical estimation of particle equilibrium distribution

As a preliminar test of our analytical results and the implementation of the simulation algorithm we numerically evaluated the equilibrium distribution of a colloid put in a harmonic trap, using the scheme illustrated in the preceeding sections. The results are presented in Fig. 2.1, and the accordance is remarkably good.

We stress that the simulation has been performed assuming a strong coupling, $\lambda = 5$, and with a field evolving on a timescale considerably slower than that of the particle, $\nu/D = 10$. In this regime the dynamical properties of the colloid are expected to be strongly influenced by the coupling, robustly corroborating our theoretical result.

Chapter 3

Dynamical properties of a trapped particle

In this Chapter we study some dynamical properties of the Brownian particle when placed in a external harmonic potential. Such a confinement can be realized experimentally via optical trapping [36]. Specifically, we derive, at lowest perturbative order in the coupling λ between the field and the particle, the autocorrelation of the probe and its average position after being released off-center. We show that these quantities decay algebraically for a critical model A field and a critical and non-critical model B field. We compute these exponent in arbitrary dimension and also show that they do not depend on the specific form of the coupling between the field and the particle.

3.1 Particle autocorrelation in the stationary state

In this section we aim to compute the probe autocorrelation

$$\langle y(t) \cdot y(t') \rangle = \sum_i \langle y_i(t) y_i(t') \rangle \tag{3.1}$$

in the stationary state and perturbatively in the coupling constant λ.

By rotational symmetry it is sufficient to consider only the autocorrelation of a single component of the position vector, i.e.

$$C(t-t') = \langle y_i(t) y_i(t') \rangle = C_0(t-t') + \lambda^2 C_2(t-t') + o(\lambda^4), \tag{3.2}$$

where the first order in λ, as well all odd orders, vanish because of the symmetry $\{\lambda, \phi\} \to \{-\lambda, -\phi\}$ of the equations of motion. Notice that the autocorrelation depends only on the difference $t-t'$ since the average is taken in the stationary state.

The autocorrelation of the uncoupled particle is (see Eq. (2.12))

$$C_0(t,t') = \langle y_i^{(0)}(t) y_i^{(0)}(t') \rangle = \frac{T}{k} e^{-\nu|t-t'|} \tag{3.3}$$

while the second order correction is

$$C_2(t,t') = \langle y_i^{(1)}(t) y_i^{(1)}(t')\rangle + \langle y_i^{(0)}(t) y_i^{(2)}(t')\rangle + \langle y_i^{(2)}(t) y_i^{(0)}(t')\rangle \tag{3.4}$$

For the first term of this expression, employing Eq. (2.20),

$$\langle y_i^{(1)}(t) y_i^{(1)}(t')\rangle = \nu^2 \int_{-\infty}^{t} \mathrm{d}s\, e^{-\nu k(t-s)} \int_{-\infty}^{t'} \mathrm{d}s'\, e^{-\nu k(t'-s')} \langle f_i^{(0)}(s) f_i^{(0)}(s')\rangle \tag{3.5}$$

where one computes

$$\langle f_i^{(0)}(s) f_i^{(0)}(s')\rangle = -\int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{\mathrm{d}^d q'}{(2\pi)^d} q_i q_i' V_q^* V_{q'}^* \langle e^{iq\cdot y^{(0)}(s)} e^{iq'\cdot y^{(0)}(s')}\rangle \langle \phi_q^{(0)}(s) \phi_{q'}^{(0)}(s')\rangle \tag{3.6}$$

Employing Eqs. (2.20) and (2.25) for the uncoupled averages of the probe and the field and carrying out the integration over the angular variables in q one gets

$$\begin{aligned}\langle y^{(1)}(t)\cdot y^{(1)}(t')\rangle = \nu^2 C_d \int_{-\infty}^{t} \mathrm{d}s\, e^{-\nu k(t-s)} \int_{-\infty}^{t'} \mathrm{d}s'\, e^{-\nu k(t'-s')} \\ \cdot \int_0^\infty \mathrm{d}q\, q^{d+1} |V_q|^2 Q_q^{\mathrm{eq}}(s-s') G_q(s-s'),\end{aligned} \tag{3.7}$$

where C_d was defined in Eq. (2.48).

For the other two terms of Eq. (3.4), using Eq. (2.39),

$$\langle y_i^{(2)}(t) y_i^{(0)}(t')\rangle = \nu \int_{-\infty}^{t} \mathrm{d}s\, e^{-\nu k(t-s)} \langle f_i^{(1)}(s) y_i^{(0)}(t')\rangle \tag{3.8}$$

Employing Eq. (2.37) for $f_i^{(1)}(s)$ and equation (2.21) for the uncoupled particle averages that appear, after some lengthy but straightforward calculations,

$$\begin{aligned}\langle y^{(2)}(t)\cdot y^{(0)}(t')\rangle = \frac{\nu C_d}{k} \int_{-\infty}^{t} \mathrm{d}s\, e^{-\nu k(t-s)} \int_{-\infty}^{s} \mathrm{d}s' \left(e^{-\nu k|t'-s'|} - e^{-\nu k|t'-s|}\right) \\ \cdot \int_0^\infty \mathrm{d}q\, q^{d+3} |V_q|^2 Q_q^{\mathrm{eq}}(s-s') G_q(s-s') \big(A_q + \nu T q^2 e^{-\nu k(s-s')}\big)\end{aligned} \tag{3.9}$$

The last term in equation (3.4) is clearly the same as this last one with t and t' exchanged.

3.1.1 Asymptotic expansion at long times

We focus here only the behaviour of $\langle y^{(1)}(t)\cdot y^{(1)}(t')\rangle$ at long time differences $t-t'$. The asymptotic behaviour for for the other terms is similar and is given in Appendix A.

By substituting $u = (t-t') - (s-s')$, $v = (t+t') - (s+s')$ and integrating over v in Eq. (3.7) one obtains

$$\langle y^{(1)}(t)\cdot y^{(1)}(t')\rangle = \frac{\nu C_d}{2k} \int_{-\infty}^{\infty} \mathrm{d}u\, e^{-\nu k|u|} \int_0^\infty \mathrm{d}q\, q^{d+1} |V_q|^2 Q_q^{\mathrm{eq}}(u-(t-t')) G_q(u-(t-t')) \tag{3.10}$$

which shows a dependence of the final result only on $t-t'$, as expected (actually on $|t-t'|$ since both $Q_q^{\mathrm{eq}}(t)$ and $G_q(t)$ depend on $|t|$ only, which is due to the $t \leftrightarrow t'$ symmetry of the quantity under consideration).

Now, without loss of generality, we expand the transform of the potential as $V_q = c_0 + c_1 q + c_2 q^2 + ...$ and accordingly its square as $|V_q|^2 = c_0^2 + c_1^2 q^2 + ...$. In doing so the autocorrelation $\langle y^{(1)}(t) \cdot y^{(1)}(t') \rangle$ can then be expressed a sum of expression identical to Eq. (3.10) but with q^n in place of V_q. In the following we therefore consider the potential $V_q = q^n$, to show the each term in this sum becomes increasingly subleading upon increasing n.

We show that only a specific u dominates the integral in Eq. (3.10) at large times. For $r > 0$, in the case of model A, substitute $u \to u/t$ and $q \to t^{1/2} q$ to obtain

$$\begin{aligned} \langle y^{(1)}(t) \cdot y^{(1)}(0) \rangle &= \frac{\nu C_d t^c}{2k} \int_{-\infty}^{\infty} \mathrm{d}u \, e^{-t(\nu k|u| + Dr|u-1|)} \\ &\qquad \cdot \int_0^\infty \mathrm{d}q \, \frac{q^{d+1+2n}}{r + q^2/t} e^{-Dq^2|u-1|} e^{-\frac{T}{k}(1-e^{-\nu kt|u-1|})q^2/t} \\ &\approx \frac{\nu C_d t^c}{2k} \int_{-\infty}^{\infty} \mathrm{d}u \, e^{-t(\nu k|u| + Dr|u-1|)} \int_0^\infty \mathrm{d}q \, \frac{q^{d+1+2n}}{r} e^{-Dq^2|u-1|} \end{aligned} \tag{3.11}$$

where the approximation follows from neglecting $o(1/t)$ terms, and c is some power that we do not need to specify. It is now clear, by the method of steepest descent, that the main contribution at large t comes from the minimum of the function $\nu k|u| + Dr|u-1|$, which is at $u = 0$ if $\nu k > Dr$, and at $u = 1$ (corresponding to $u = t$ in Eq. (3.10)), if $\nu k < Dr$. In the case of model B, with the same substitutions as before, and neglecting again $o(1/t)$ terms, one finds

$$\langle y^{(1)}(t) \cdot y^{(1)}(0) \rangle \approx \frac{\nu C_d t^c}{2k} \int_{-\infty}^{\infty} \mathrm{d}u \, e^{-\nu kt|u|} \int_0^\infty \mathrm{d}q \, \frac{q^{d+1+2n}}{r} e^{-Drq^2|u-1|} \tag{3.12}$$

which shows that the dominant contribution is always from $u = 0$ (c is again an inessential power). In the case $r = 0$, both for model A and model B, a very similar reasoning indicates that the dominant contribution is always from $u = 0$.

For a field obeting model A dynamics, in the case $Dr > \nu k$, we can immediately conclude that $\langle y^{(1)}(t) \cdot y^{(1)}(0) \rangle \sim e^{-\nu kt}$. For all other cases instead, both for model A and B, the integral is then asymptotic to the integrand evaluated in $u = 0$

$$\langle y^{(1)}(t) \cdot y^{(1)}(0) \rangle \sim \int_0^\infty \mathrm{d}q \, q^{d+1+2n} Q_q^{\mathrm{eq}}(t) G_q(t) \tag{3.13}$$

For a non-critical model A field

$$\begin{aligned}\langle y^{(1)}(t)\cdot y^{(1)}(0)\rangle &\sim e^{-Drt}\int_0^\infty \mathrm{d}q\, q^{d+1+2n} e^{-\frac{T}{k}(1-e^{-\nu k t})q^2}\frac{1}{r+q^2}e^{-Dtq^2} \\ &\sim e^{-Drt}\int_0^\infty \mathrm{d}q\,\frac{q^{d+1+2n}}{r+q^2}e^{-Dtq^2} \\ &\sim e^{-Drt}t^{-(d/2+n+1)}\int_0^\infty \mathrm{d}p\,\frac{p^{d+1+2n}}{r+\underbrace{p^2/t}_{\to 0}}e^{-Dq^2} \\ &\sim e^{-Drt}\end{aligned} \tag{3.14}$$

where $p = t^{1/2}q$. For a critical model A field with a similar reasoning we get instead

$$\langle y^{(1)}(t)\cdot y^{(1)}(0)\rangle \sim t^{-(d/2+n)} \tag{3.15}$$

For a non-critical model B field

$$\begin{aligned}\langle y^{(1)}(t)\cdot y^{(1)}(0)\rangle &\sim \int_0^\infty \mathrm{d}q\,\frac{q^{d+1+2n}}{r+q^2}e^{-Dt(r+q^2)q^2} \\ &\sim t^{-(d/2+1+n)}\int_0^\infty \mathrm{d}p\,\frac{p^{d+1+2n}}{r+\underbrace{p^2/t}_{\to 0}}e^{-D(r+\underbrace{p^2/t}_{\to 0})p^2} \\ &\sim t^{-(d/2+1+n)}\end{aligned} \tag{3.16}$$

where $p = t^{1/2}q$, while for a critical model B field

$$\langle y^{(1)}(t)\cdot y^{(1)}(0)\rangle \sim \int_0^\infty \mathrm{d}q\, q^{d-1+2n}e^{-Dtq^4} \sim t^{-(d/4+n/2)} \tag{3.17}$$

with $p = t^{1/4}q$.

As anticipated, these results show that the asymptotic of the autocorrelation is increasingly subleading upon increasing the order n of the potential, which is illustrated in the left panels of Figs. 3.3 and 3.6. The leading contribution at long is times is then given by the $n = 0$ term, which we suppose to be non-vanishing.

To summarize, the asymptotic behaviour for the autocorrelation of a particle trapped in a harmonic potential, coupled to a Gaussian field obeying model A dynamics, does not depend on the details of this coupling, and is

$$\langle y(t)\cdot y(0)\rangle \sim \lambda^2\begin{cases} e^{-\nu k t}, & Dr > \nu k, \\ e^{-Drt}, & Dr < \nu k, \\ t^{-d/2}, & r = 0.\end{cases} \tag{3.18}$$

which is presented in Figs. 3.1, 3.2 and 3.3.

If the field evolves according to model B dynamics instead the asymptotic behaviour of the autocorrelation function of the position of the trapped probe is

$$\langle y(t) \cdot y(0) \rangle \sim \lambda^2 \begin{cases} t^{-(d/2+1)}, & r > 0, \\ t^{-d/4}, & r = 0, \end{cases} \tag{3.19}$$

which is plotted in Figs. 3.4, 3.5 and 3.6.

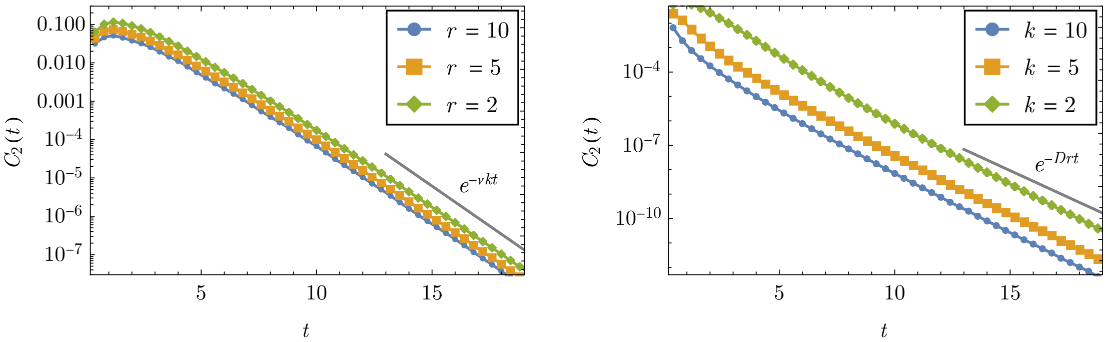

Figure 3.1: Exponential decay of the temporal autocorrelation of the particle position for a non-critical ($r > 0$) model A field, for various values of the relevant parameters. Data points are obtained from the numerical integration Eqs. (3.7,3.9). Solid gray lines are the expected exponential decay at long times, Eq. (3.18). For both figures $d = 1$, the field-particle interaction is point-like i.e. $V(x) = \delta(x)$, and $T, D, \nu = 1$. In the left panel $k = 1$, in the right panel $r = 1$.

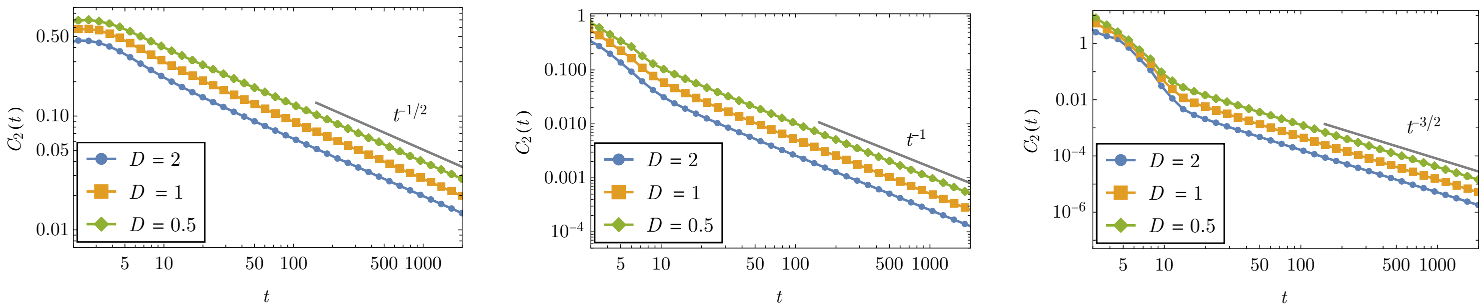

Figure 3.2: Power-law decay of the autocorrelation of the particle position for a critical ($r = 0$) model A field, in the case $d = 1$ (left), $d = 2$ (center) and $d = 3$ (right). The field-particle interaction is point-like. Data points are obtained from the numerical integration of Eqs. (3.7,3.9). Solid gray lines are the expected $t^{-d/2}$ power-law decay. In all three figures $T, D, \nu, k = 1$.

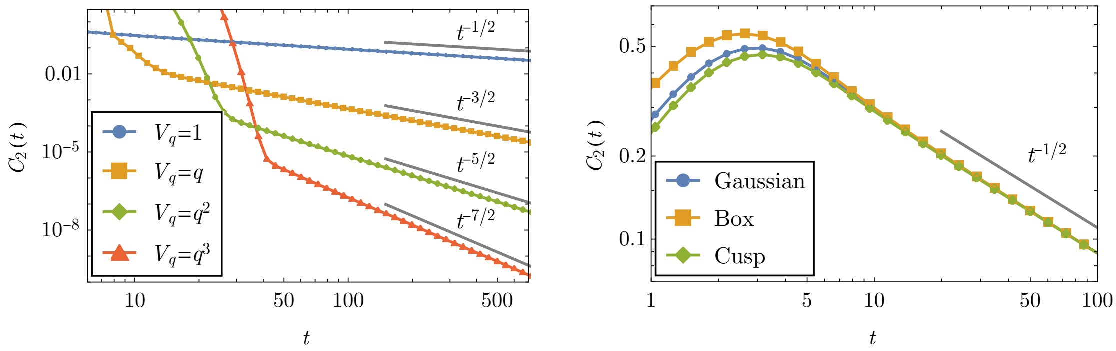

Figure 3.3: Power-law decay of the autocorrelation of the particle position for various type of field-particle interactions, in the case of a critical ($r = 0$) model A field. Data points are obtained from the numerical integration Eqs. (3.7,3.9). The left panel shows the power-law decay $t^{-(d/2+n)}$ for the potentials $V_q = q^n$. As discussed in the text, this implies that any potential with $V_{q=0} \neq 0$ will have the same long-time behaviour, which is exemplified in the right panel. In this panel the potentials are $V_q = e^{-\varepsilon^2 q^2/2}$ (gaussian), $V_q = \mathrm{sinc}(\varepsilon q/2)$ (box) and $V_q = 1/(1 + \varepsilon^2 q^2)$ (cusp) respectively. In both panels $d = 1$, $T, D, \nu, k = 1$, and $\varepsilon = 0.5$.

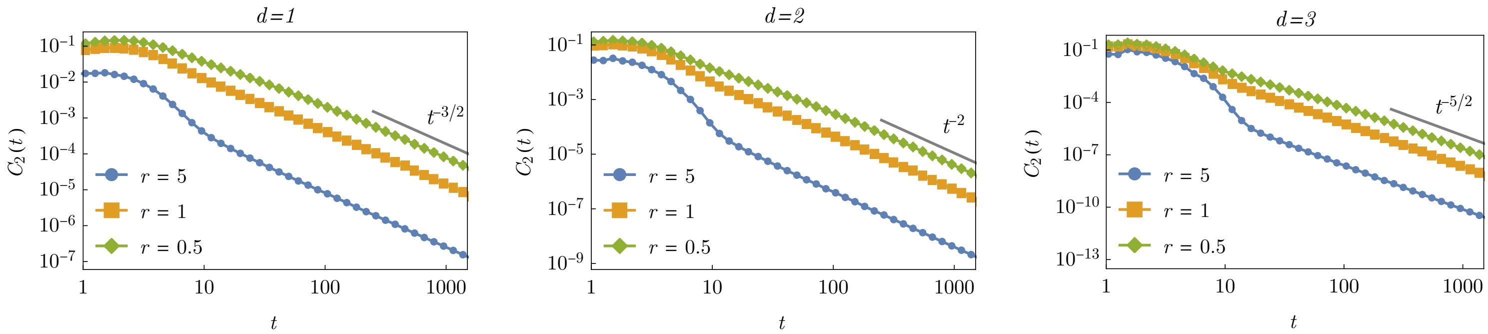

Figure 3.4: Power-law decay of the autocorrelation of the particle position for a non-critical model B field, in $d = 1$ (left), $d = 2$ (center) and $d = 3$ (right) dimensions. The field-particle interaction is point-like. Data points are obtained as a numerical integration of equations (3.7,3.9). Solid gray lines are the expected $t^{-(d/2+1)}$ power-law decay. In all three panels $T, D, \nu, k = 1$.

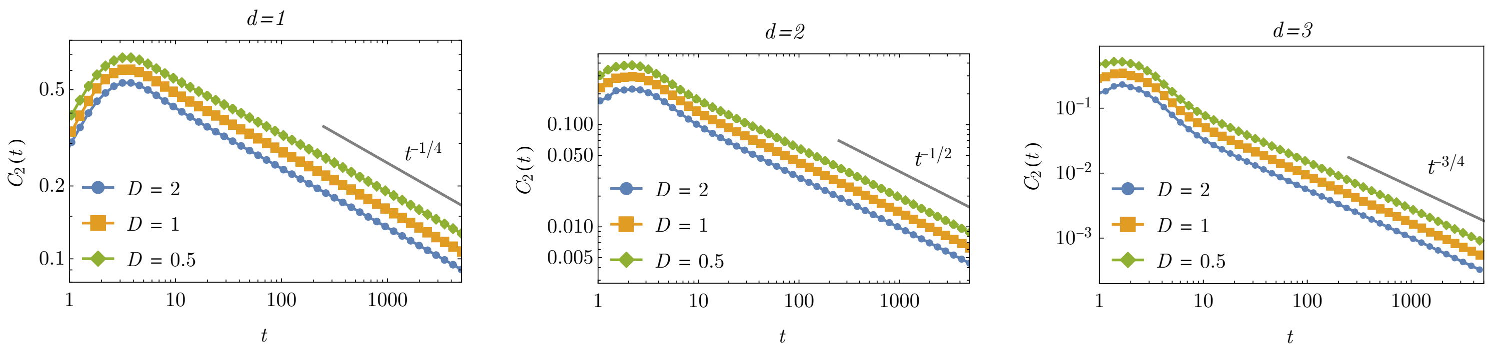

Figure 3.5: Power-law decay of the autocorrelation of the particle position for a critical ($r = 0$) model B field, in $d = 1$ (left), $d = 2$ (center) and $d = 3$ (right) dimensions. The field-particle interaction is point-like. Data points are obtained as a numerical integration of equations (3.7,3.9). Solid gray lines are the expected $t^{-d/4}$ power-law decay. In all three panels $T = \nu = k = 1$.

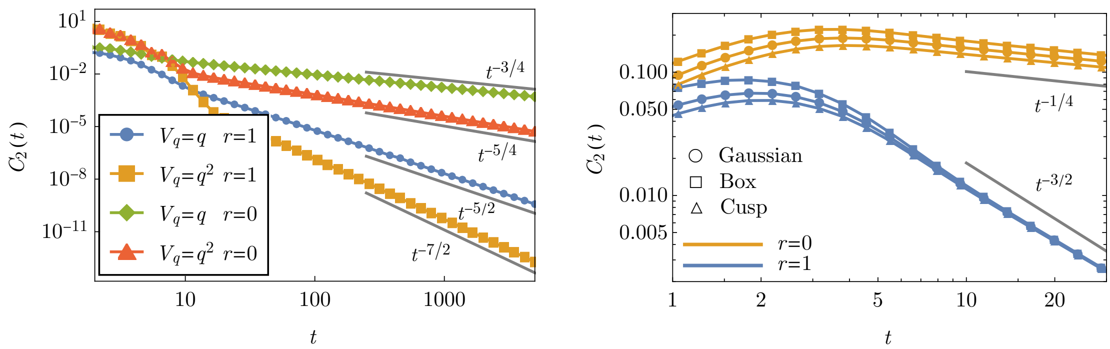

Figure 3.6: Power-law decay of the autocorrelation position of a probe coupled to the field with various type of coupling to a model B field. Data points are obtained as a numerical integration of equations (3.7,3.9). The left panel shows the power-law decay $t^{-(d/2+n+1)}$ and $t^{-(d/4+n/2)}$ for the potentials $V_q = q^n$. The right panel shows the asymptotic behavior for the potential $V_q = e^{-\varepsilon^2 q^2/2}$ (gaussian), $V_q = \mathrm{sinc}(\varepsilon q/2)$ (box) and $V_q = 1/(1 + \varepsilon^2 q^2)$ (cusp). In both figures $d = 1$, $T, D, \nu, k = 1$, and $\varepsilon = 0.5$.

3.2 Relaxation towards equilibrium

Another interesting issue is whether the coupling between the field and the particle modifies the relaxation behaviour of the latter when initially out of equilibrium. We address this problem by analyzing the simple scenario in which the colloid is initially released off-center, and study its average position.

By the rotational symmetry of the system we can take the initial position of the particle y_0 to be non-zero only along one axis e.g. $\vec{y}_0 = y_0\hat{x}$. Moreover, again by rotational symmetry, the average position of the particle will remain non-zero only along this axis. We therefore refer to the average position as $\langle y(t)\rangle$ without specifying its spatial direction.

In order to investigate the effect of the coupling to the field we again proceed perturbatively in the coupling strength λ. The equations of motion at each order are the same in the previous section, with the important difference that here we .are not considering the stationary state.

At the first order trivially $\langle y^{(1)}(t)\rangle = 0$ from Eqs. (2.38) and (2.36), in accordance to the symmetry $\{\lambda, \phi\} \to \{-\lambda, -\phi\}$.

To compute $\langle y^{(2)}(t)\rangle$ we need the average $\langle f_i^{(1)}(s)\rangle = F_1 + F_2$, where

$$\begin{aligned} F_1 &= \langle\Big[\int \mathrm{d}^d x\, \phi^{(0)}(x,s)\partial_i\partial_j V(x - y^{(0)}(s))\Big]\Big[y_j^{(1)}(s)\Big]\rangle \\ &= -\nu\int_0^s \mathrm{d}s'\, e^{-\nu k(s-s')}\int \frac{\mathrm{d}^d q}{(2\pi)^d}\int\frac{\mathrm{d}^d q'}{(2\pi)^d} i q_i q_j q'_j V_q^* V_{q'}^* \\ &\qquad\qquad \cdot \langle e^{iq\cdot y^{(0)}(s)} e^{iq'\cdot y^{(0)}(s')}\rangle\langle\phi_q^{(0)}(s)\phi_{q'}^{(0)}(s')\rangle \\ &= i\nu\int_0^s \mathrm{d}s'\, e^{-\nu k(s-s')}\int\frac{\mathrm{d}^d q}{(2\pi)^d} q_i q^2 |V_q|^2 Q_q(s,s') G_q(s-s') \end{aligned} \tag{3.20}$$

and

$$F_2 = -\langle\Big[\int \mathrm{d}^d x\, \phi^{(1)}(x,s)\partial_i V(x - y^{(0)}(s))\Big]\rangle \tag{3.21}$$

$$= iD\int_0^s \mathrm{d}s' \int\frac{\mathrm{d}^q q}{(2\pi)^d} q_i q^\alpha |V_q|^2 Q_q(s,s') e^{-A_q(s-s')} \tag{3.22}$$

Accordingly, the second-order correction to the average of the particle position is

$$\begin{aligned} \langle y^{(2)}(t)\rangle = \frac{\nu}{T}\int_0^t \mathrm{d}s\, e^{-\nu k(t-s)}\int_0^s \mathrm{d}s' \int\frac{\mathrm{d}^d q}{(2\pi)^d} q_i |V_q|^2 \sin\big(q_i y_i(0)(e^{-\nu k s'} - e^{-\nu k s})\big) \\ \cdot\, e^{-R(s,s')q^2} G_q(s-s')\big(A_q + \nu T q^2 e^{-\nu k(s-s')}\big) \end{aligned} \tag{3.23}$$

which is the main result of this section. Differently from the previous section we can not factorize the integral over the angular variables.

3.2.1 Asymptotic expansion at long times

In order to determine the long-time behaviour of the relaxation we first identify the dominant contribution in the time integrals in Eq. (3.23). In a similar way done in the previous section, one can rescale $s \to s/t$, $s' \to s'/t$ and $q \to t^{1/2}q$ to establish that the portion of the integrand that gives the dominating values in the time integrals is, in the case of a model A field,

$$e^{\nu ks}e^{-Dr(s-s')}\sin\big(qy(0)(e^{-\nu ks'}-e^{-\nu ks})\big), \tag{3.24}$$

which is trivially proportional to

$$(e^{\nu k(s-s')}-1)e^{-Dr(s-s')}\,\mathrm{sinc}(qy(0)(e^{-\nu ks'}-e^{-\nu ks})). \tag{3.25}$$

From this last expression it is clear that if $Dr > \nu k$ the dominant contribution is from the region $s = s'$, while if $Dr < \nu k$ from the point $s = t$, $s' = 0$. In the case of a model B dynamics for the field, instead, the dominating values are always $s = t$, $s' = 0$, the difference being the absence of the $e^{-Dr(s-s')}$ term in the previous equations.

For model A and $Dr > \nu k$ we proceed by expanding in the difference $s - s'$ and then substituting $u = s - s'$, $v = s + s'$. By doing so, one straightforwardly obtains $y^{(2)}(t) \sim e^{-\nu kt}$.

For model A and $Dr < \nu k$ instead the integral is simply asymptotic to the integrand evaluated in $s = t$, $s' = 0$

$$\langle y^{(2)}(t)\rangle \sim \int \mathrm{d}^d q \sin\Big(q_i y_0(1-e^{-\nu kt})\Big)q_i q^{2n}e^{-R(t,0)}e^{-D(r+q^2)t}\frac{1}{r+q^2}(A_q+\nu Tq^2e^{-\nu kt}). \tag{3.26}$$

Neglecting $o(1/t)$ terms and expanding the sine as $\sin x = \sum_m c_m x^{2m+1}$, one finds,

$$\begin{aligned}\langle y^{(2)}(t)\rangle &\sim \int \mathrm{d}^d q \sin(q_i y_0)q_i q^2 n e^{-Tq^2/2k}e^{-D(r+q^2)t}\\ &= e^{-Drt}\sum_m c_m\int_0^\infty \mathrm{d}q\, q_i^{2m+2}q^{2n}e^{-(T/2k+Dt)q^2}\\ &= e^{-Drt}\sum_m c_m t^{-(d/2+1+n+m)}\end{aligned} \tag{3.27}$$

where c_m are some constants that we do not need to specify. Each term in the sum is increasingly more subleading, so we get that $y^{(2)}(t) \sim e^{-Drt}$ if $0 < Dr < \nu k$ and as $y^{(2)}(t) \sim t^{-(d/2+1+n)}$ if $r = 0$.

For model B, in a similar way

$$\langle y^{(2)}(t)\rangle \sim \sum_m c_m\int_0^\infty \mathrm{d}q\, q_i^{2m+4}q^{2n}e^{-(T/2k+Drt)q^2-Dtq^4}, \tag{3.28}$$

which is asymptotic to $t^{-(d/2+2+n)}$ for $r > 0$ and to $t^{-(d/4+1+n/2)}$ for $r = 0$.

To summarize, the average position of a harmonically trapped colloid coupled to a field obeying model A dynamics initially released off-center relaxates asymptotically as

$$\langle y(t)\rangle \sim \lambda^2 \begin{cases} e^{-\nu k t}, & Dr > \nu k, \\ e^{-Drt}, & Dr < \nu k, \\ t^{-(d/2+1)}, & r = 0 \end{cases} \tag{3.29}$$

which is presented in Figs. 3.7, 3.8 and 3.9.

If the field follows model B dynamics instead

$$\langle y(t)\rangle \sim \lambda^2 \begin{cases} t^{-(d/2+2)} & r > 0 \\ t^{-(d/4+1)} & r = 0 \end{cases} \tag{3.30}$$

which is plotted in Figs. 3.10 and 3.11.

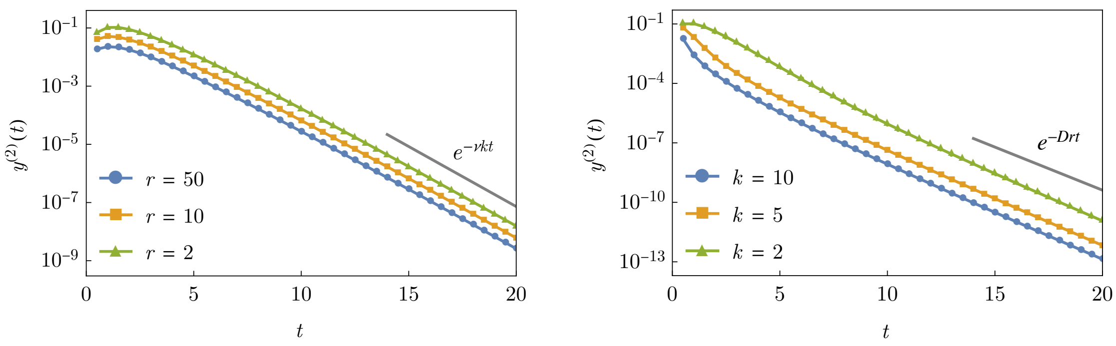

Figure 3.7: Theoretical result for the relaxation of a harmonically trapped probe, coupled to a non-critical ($r > 0$) model A field, initially released off-center. The figures show the second-order correction, in the particle-field coupling strength λ, to the average position of the particle. Points are obtained by numerically integrating equation (3.23). In the left panel $Dr > \nu k$, while in the right panel $Dr < \nu k$. Solid gray lines are the expected exponential decay at long times, equation (3.29). For both figures $d = 1$, the field-particle interaction is point-like i.e. $V(x) = \delta(x)$ and $T, D, \nu, y(0) = 1$. In the left panel $k = 1$, while in the right panel $r = 1$.

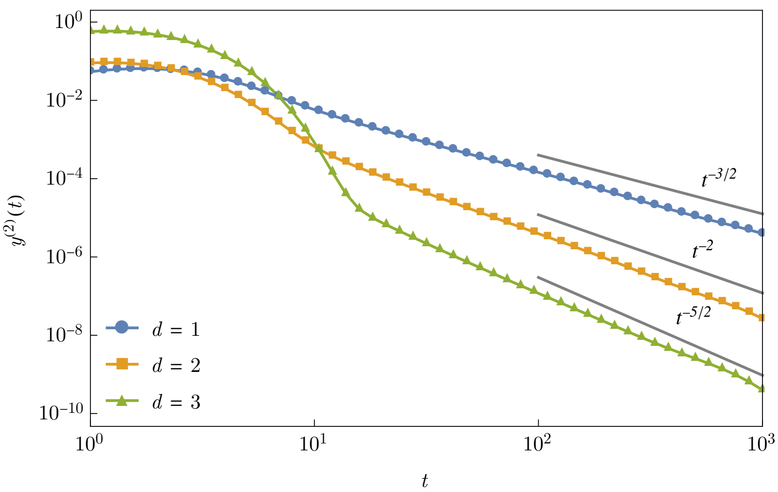

Figure 3.8: Theoretical result for the relaxation of a harmonically trapped probe coupled to a critical ($r = 0$) model A field for different spatial dimensions d. Data points are obtained by numerically integrating equation (3.23). Solid gray lines are the expected power-law decay at long times, Eq. (3.29). The field-particle interaction is point-like and $T, D, \nu, k, y(0) = 1$.

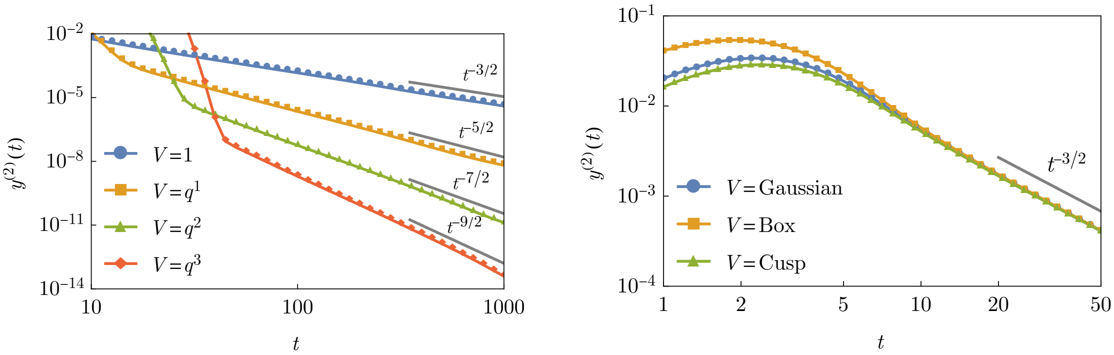

Figure 3.9: Analysis of the dependence of the relaxation of a probe coupled to a critical ($r = 0$) model A field on the field-particle interactions V. Points are obtained by numerically integrating equation (3.23). In the left panel solid gray lines are the expected $t^{-(d/2+1+n)}$ power-law decay for the interaction $V_q = q^n$. As discussed in the text, this implies that any potential with non-zero $V_{q=0}$ will have the same long-time behaviour. This is exemplified in the right panel where the interactions are $V_q = e^{-\varepsilon^2 q^2/2}$ (gaussian), $V_q = \text{sinc}(\varepsilon q/2)$ (box) and $V_q = 1/(1 + \varepsilon^2 q^2)$ (cusp). In both panels $d = 1$, $T, \nu, k, y(0) = 1$ and $\varepsilon = 0.5$.

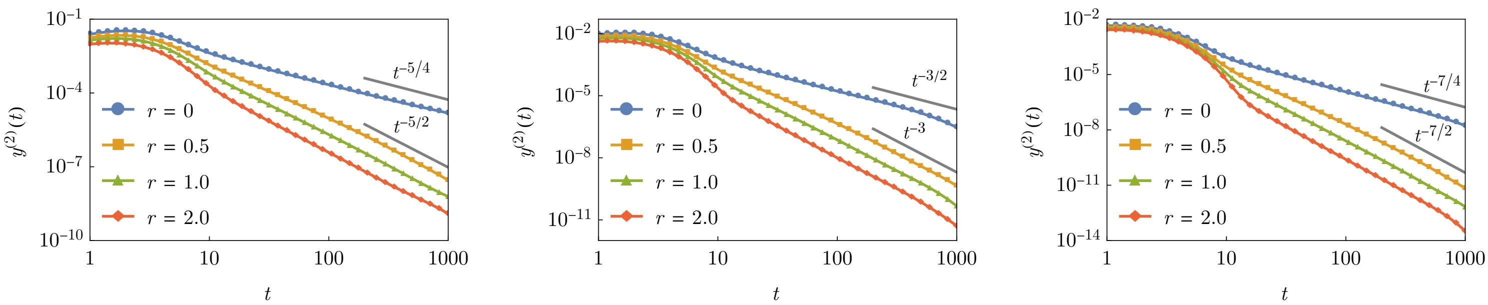

Figure 3.10: Theoretical result for the relaxation of a harmonically trapped probe coupled to a model B field for $d = 1$ (left), $d = 2$ (middle) and $d = 3$ (right). Data points are obtained by numerically integrating equation (3.23). Solid gray lines are the expected power-law decay at long times, Eq. (3.30). The field-particle interaction is point-like and $T, D, \nu, k, y(0) = 1$.

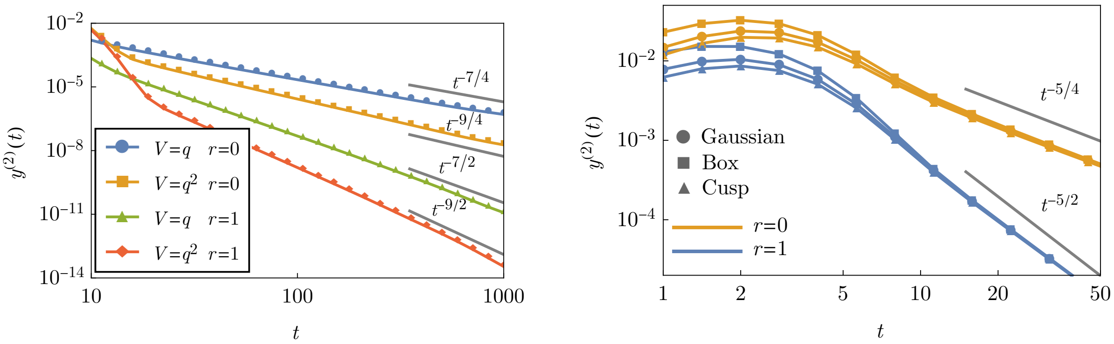

Figure 3.11: Dependence of the relaxation of a probe coupled to a model B field on the field-particle interactions V. Points are obtained by numerically integrating equation (3.23). In the left panel solid gray lines are the expected $t^{-(d/2+2+n)}$ $(r = 0)$ and $t^{-(d/4+1+n/2)}$ $(r > 0)$ power-law decay for the interaction $V_q = q^n$, equation. As discussed in the text, this implies that any potential with non-zero $V_{q=0}$ will have the same long-time behaviour. This is exemplified in the right panel where the interactions are $V_q = e^{-\varepsilon^2 q^2/2}$, $V_q = \mathrm{sinc}(\varepsilon q/2)$ and $V_q = 1/(1 + \varepsilon^2 q^2)$ respectively. In both panels $d = 1$, $T, \nu, k, y(0) = 1$ and $\varepsilon = 0.5$.

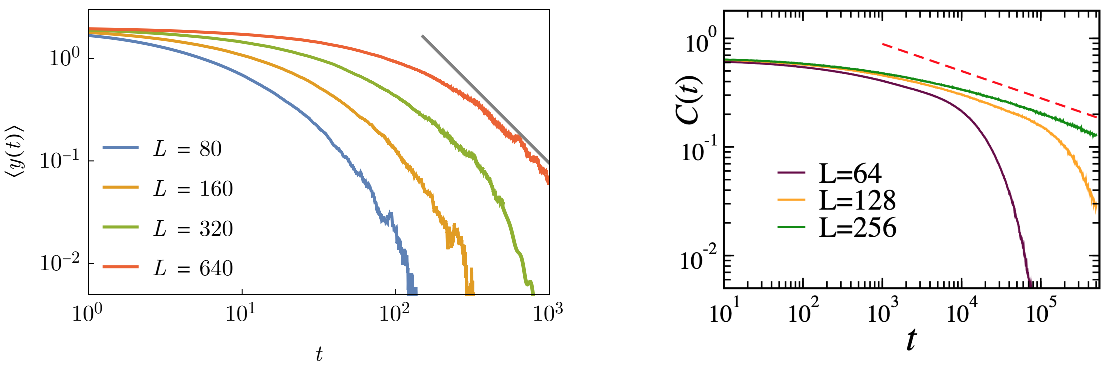

Figure 3.12: Numerical analysis of the relaxation in a model A field (left) and autocorrelation in a model B field (right) of a tracer particle. Both simulation have been performed with the methods discussed in section 2.7. The right panel shows the results of a simulation performed by U. Basu with the same code developed during the work this thesis and privately communicated to the author. For both simulations are $T, D, \nu, k = 1$, $r = 0$, $\Delta t = 0.01$ and $\lambda = 0.5$. In the left panel the field follows model A dynamics, $\Delta x = 0.2$ and $y(0) = 2$, while in the right panel the field follows model B dynamics and $\Delta x = 1$.

3.3 Numerics

The previous results on the autocorrelation in the stationary state of a probe coupled to a critical fluid and its relaxation towards equilibrium have been numerically investigated using the simulation algorithm presented in Section 2.7. The results are presented in Fig. 3.12.

The relaxational behaviour was simulated in a critical model A field. After an initial thermalization of the field, the particle was put a number of lattice sites away from the center of the trap. Then the particle and the field were evolved using the scheme discussed in Section 2.7 and the trajectory of the first recorded. This procedure was repeated around 10^5 - 10^6 times to get the average value of the position of the particle at each timestep.

Since the simulation takes place in a finite system of size L the correlation length at the transition point is effectively finite and of order $\xi_{\text{eff}} \sim L$ or, equivalently, the parameter r denoting the deviation from criticality is effectively non-zero and of order $r_{\text{eff}} \sim L^{-2}$. Thus, the simulation has been repeated by increasing the size of the system to try to infer the limiting behaviour in an infinite system. In the left panel of Fig. 3.12 the result are presented together with the predicted $t^{-3/2}$ power-law. The results, while not completely definitive, surely indicate that the transition influences strongly the dynamical behavior of the colloid and a transition to an algebraic decay seems to be taking place.

The autocorrelation of the particle was obtained numerically in a similar way, with the important difference that in this case the simulated field was taken to follow model B dynamics. Also, since the autocorrelation was derived in the stationary state analytically, data was collected after a brief thermalization period of both the field an the particle. In this case the results in Fig. 3.12 give a stronger evidence of the predicted $t^{-1/4}$ power-law decay.

Overall, these numerical result seem to indicate that the weak-coupling theoretical results derived in this chapter are apt to describe more realistic non-perturbative scenarios.

3.4 Discussion

In hindsight, the asymptotic behaviour of the dynamical quantities analyzed in this Chapter could have been derived, at least qualitatively, from heuristic arguments. Indeed, they all follow by simply assuming that in the presence of the coupling the whole dynamics is limited by the slowest scale in the system.

To be definite, consider a non-critical model A field. The timescale on which the particle is $1/\nu k$, while that of each mode of the field is finite and equal to $1/D(r + q^2)$. The slowest evolving modes are long-wavelength ones, whose timescale is $1/Dr$ in the limit $q \to 0$. Therefore the larger timescale of the coupled system of the colloid and the field is given by the larger between $1/\nu k$ and $1/Dr$. The timescale limiting the dynamics is then $1/\nu k$ when $\nu k < Dr$, and $1/Dr$ when $\nu k > Dr$. This is exactly what was found for both the autocorrelation and the relaxation, Eqs. (3.18) and (3.29).

Approaching the critical point in model A dynamics, the critical slowing down of the order parameter, due to its the diverging timescales, render the dynamical behaviour of the colloid algebraic. In the case of model B the time scale of the $q = 0$ mode is always diverging, explaining the power-law found in the autocorrelation and relaxation of the particle both in the critical and non-critical case.

Unfortunately, this simple argument alone cannot predict the specific exponents of these power-laws or their dependence on dimensionality, and a more quantitative calcuation as the one present in this Chapter was needed.

We conclude by noticing that all the exponents derived satisfy the relation

$$\langle y(t) \rangle \sim t^{-1} \langle y(t) \cdot y(0) \rangle. \tag{3.31}$$

which would is just a fluctuation-dissipation relation, connecting the relaxation of the particle with its correlations at equilibrium. At present, however, we are not able to give a more precise statement of this relation, in particular if this relation holds also outside the perturbative regime.

Chapter 4

Cross-correlation of two particles

In this Chapter we present some preliminar analytical results for the correlations between two Brownian particles, with no direct interparticle interaction, but that are simultaneously interacting with the fluctuating field.

4.1 Equations of motion for two particles

We refer to the coordinates of the two particle with y and z. We take the Hamiltonian of the system to be

$$H = H_\phi[\phi] + \frac{1}{2}k_y y^2 + \frac{1}{2}k_z z^2 - \lambda \int \mathrm{d}^d x\, \phi(x) \left[V_y(x-y) + V_z(x-z)\right], \tag{4.1}$$

which is a trivial generalization of the one used in the previous Chapters. We stress that with this Hamiltonian the particles do not interact directly. Any effect of the motion of one on the other can be regarded as an effective interaction mediated by the field.

The equations of motion for the field and the two particles are, with this Hamiltonian,

$$\partial_t \phi(x,t) = -A\phi(x,t) + \lambda D(i\nabla)^\alpha (V_y(x-y(t)) + V_z(x-z(t))) + \zeta(x,t), \tag{4.2}$$

$$\dot{y}(t) = -\nu_y k_y y(t) + \lambda \nu_y f_y(t) + \xi_y(t), \tag{4.3}$$

$$\dot{z}(t) = -\nu_z k_z z(t) + \lambda \nu_z f_z(t) + \xi_z(t), \tag{4.4}$$

where we introduced the forces

$$\begin{aligned} f_y(t) &= \nabla_y \int \mathrm{d}^d x\, \phi(x,t) V_y(x-y(t)), \\ f_z(t) &= \nabla_z \int \mathrm{d}^d x\, \phi(x,t) V_z(x-z(t)). \end{aligned} \tag{4.5}$$

All the noises have zero mean and their correlations are explicitly

$$\begin{aligned}\langle\zeta(x,t)\zeta(x',t')\rangle &= 2DT(i\nabla)^{\alpha}\delta(x-x')\delta(t-t') \\ \langle\xi_{y,i}(t)\xi_{y,j}(t')\rangle &= 2\nu_y T\delta_{ij}\delta(t-t') \\ \langle\xi_{z,i}(t)\xi_{z,j}(t')\rangle &= 2\nu_z T\delta_{ij}\delta(t-t')\end{aligned} \tag{4.6}$$

4.2 Derivation of the cross-correlation

In order to highlight the effect of the interaction induced between the two particles by the presence of the field we aim to compute the stationary-state cross-correlation

$$\langle y(t)\cdot z(t')\rangle = \sum_i \langle y_i(t) z_i(t')\rangle \tag{4.7}$$

In the absence of effective interaction, the coordinates of these two particles are independent and therefore the average above factorizes and vanishes, since $\langle y_i(t)\rangle = \langle z_i(t')\rangle = 0$.

4.2.1 Perturbative expansion

The weak-coupling approximation employed in the previus chapter can be easily generalized to the present situation.

At lowest order in the coupling λ between the field and each of the two particles the equations of motion are

$$\begin{aligned}\dot{y}^{(0)}(t) &= -\nu_y k_y y^{(0)}(t) + \xi_y(t), \\ \dot{z}^{(0)}(t) &= -\nu_z k_z z^{(0)}(t) + \xi_z(t),\end{aligned} \tag{4.8}$$

which are solved by the Ornstein-Uhlenbeck process. The equation for the field remains the same as before, Eq. (2.22).

At order λ^1, instead, we find for the particles

$$\begin{aligned}\dot{y}^{(1)}(t) &= -\nu_y k_y y^{(1)}(t) + \nu_y f_y^{(0)}(t), \\ f_{y,i}^{(0)}(s) &= \int \frac{\mathrm{d}^d q}{(2\pi)^d} e^{iq\cdot y^{(0)}(s)} iq_i \phi_q^{(0)}(s) V_{y,q}^*,\end{aligned} \tag{4.9}$$

and

$$\begin{aligned}\dot{z}^{(1)}(t) &= -\nu_z k_z z^{(1)}(t) + \nu_z f_z^{(0)}(t), \\ f_{z,i}^{(0)}(s) &= \int \frac{\mathrm{d}^d q}{(2\pi)^d} e^{iq\cdot z^{(0)}(s)} iq_i \phi_q^{(0)}(s) V_{z,q}^*,\end{aligned} \tag{4.10}$$

which can be solved with

$$\begin{aligned}y^{(1)}(t) &= \nu_y \int_{-\infty}^{t} \mathrm{d}s\, e^{-\nu_y k_y (t-s)} f_y^{(0)}, \\ z^{(1)}(t) &= \nu_z \int_{-\infty}^{t} \mathrm{d}s\, e^{-\nu_z k_z (t-s)} f_z^{(0)}(s).\end{aligned} \tag{4.11}$$

The equation of motion of the field and its solution at this order are

$$\partial_t \phi^{(1)}(x,t) = -A\phi^{(1)}(x,t) + D(i\nabla)^\alpha (V_y(x - y^{(0)}(t)) + V_z(x - z^{(0)}(t))), \tag{4.12}$$

$$\phi_q^{(1)}(t) = Dq^\alpha \int_{-\infty}^{t} \mathrm{d}s\, e^{-A_q(t-s)} (V_{y,q} e^{-iq\cdot y^{(0)}(s)} + V_{z,q} e^{-iq\cdot z^{(0)}(s)}). \tag{4.13}$$

At order λ^2 the equations for the particles are

$$\begin{aligned} y_i^{(2)}(t) &= \nu_y \int_{-\infty}^{t} \mathrm{d}s\, e^{-\nu_y k_y (t-s)} f_{y,i}^{(1)}(s), \\ f_{y,i}^{(1)}(s) &= \int \frac{\mathrm{d}^d q}{(2\pi)^d} e^{iq\cdot y^{(0)}(s)} (iq_i \phi_q^{(1)}(s) - q_i q_k \phi_q^{(0)}(s) y_k^{(1)}(s)) V_{y,q}^*, \end{aligned} \tag{4.14}$$

and

$$\begin{aligned} z_i^{(2)}(t) &= \nu_z \int_{-\infty}^{t} \mathrm{d}s\, e^{-\nu_z k_z (t-s)} f_{z,i}^{(1)}(s), \\ f_{z,i}^{(1)}(s) &= \int \frac{\mathrm{d}^d q}{(2\pi)^d} e^{iq\cdot z^{(0)}(s)} (iq_i \phi_q^{(1)}(s) - q_i q_j \phi_q^{(0)}(s) z_j^{(1)}(s)) V_{z,q}^*, \end{aligned} \tag{4.15}$$

4.2.2 Particles correlations

With the previous result the cross-correlation between the two particles can be obtained in a conceptually similar way as it as been done in deriving similar quantities in this thesis, so we just present the final results.

The correlations of the forces are

$$\langle f_{y,i}^{(0)}(s) f_{z,i}^{(0)}(s') \rangle = \int \frac{\mathrm{d}^d q}{(2\pi)^d} q_i^2 V_{y,q} V_{z,q}^* G_q(s-s') e^{-Tq^2/2k_y} e^{-Tq^2/2k_z}. \tag{4.16}$$

The cross-correlations are

$$\begin{aligned} \langle y^{(1)}(t)\cdot z^{(1)}(t') \rangle = \nu_y \nu_z C_d &\int_{-\infty}^{t} \mathrm{d}s\, e^{-\nu_y k_y (t-s)} \int_{-\infty}^{t'} \mathrm{d}s'\, e^{-\nu_z k_z (t'-s')} \\ &\cdot \int_0^\infty \mathrm{d}q\, q^{d+1} V_{y,q} V_{z,q}^* G_q(s-s') e^{-Tq^2/2k_y} e^{-Tq^2/2k_z}, \end{aligned} \tag{4.17}$$

$$\begin{aligned} \langle y^{(2)}(t)\cdot z^{(0)}(t') \rangle = \frac{\nu_y D T C_d}{k_z} &\int_{-\infty}^{t} \mathrm{d}s \int_{-\infty}^{s} \mathrm{d}s'\, e^{-\nu_y k_y (t-s)} \\ &\cdot \int_0^\infty q^{d+1} q^\alpha e^{-A_q(s-s')} V_{y,q} V_{z,q}^* e^{-\nu_z k_z |t'-s'|} e^{-Tq^2/2k_y} e^{-Tq^2/2k_z}, \end{aligned} \tag{4.18}$$

$$\begin{aligned} \langle y_i^{(0)}(t) z_j^{(2)}(t') \rangle = \frac{\nu_z D T C_d}{k_y} &\int_{-\infty}^{t'} \mathrm{d}s \int_{-\infty}^{s} \mathrm{d}s'\, e^{-\nu_z k_z (t'-s)} \\ &\cdot \int_0^\infty q^{d+1} q^\alpha e^{-A_q(s-s')} V_{y,q} V_{z,q}^* e^{-\nu_y k_y |t-s'|} e^{-Tq^2/2k_y} e^{-Tq^2/2k_z}. \end{aligned} \tag{4.19}$$

where C_d was defined in Eq. (2.48).

4.3 Asymptotic expansion at long times

We consider here the particular case in which the two particles are identical and the trapped in the same potential. This corresponds to setting $\nu_y = \nu_z = \nu$, $k_y = k_z = k$, $V_y = V_z = V$.

In order to obtain the expansion of $\langle y^{(1)}(t) \cdot z^{(1)}(t') \rangle$ one proceeds similarly as done with the leading term in the autocorrelation, Eq. (3.7). One gets, for model A,

$$\langle y^{(1)}(t) \cdot z^{(1)}(0) \rangle \sim \begin{cases} e^{-\nu k t} & \text{for} \quad Dr > \nu k, \\ e^{-Drt} & \text{for} \quad Dr < \nu k, \\ t^{-(d/2+n)} & \text{for} \quad r = 0, \end{cases} \tag{4.20}$$

while for model B

$$\langle y^{(1)}(t) \cdot z^{(1)}(0) \rangle \sim \begin{cases} t^{-(d/2+n+1)} & \text{for} \quad r > 0, \\ t^{-(d/4+n/2)} & \text{for} \quad r = 0. \end{cases} \tag{4.21}$$

Here, n is the order of the potential $V_q = q^n$, introduced for the same reasons seen in Chapter 2.

For the term $\langle y^{(2)}(t) \cdot z^{(0)}(t) \rangle$ start by shifting $s \to s - t'$, $s' \to s' - t'$ to get an expression that depends solely on $t - t'$, set $t' = 0$, and consider $V_q = q^n$ again

$$\begin{aligned} \langle y^{(2)}(t) \cdot z^{(0)}(0) \rangle = \frac{\nu D T C_d}{k} \int_{-\infty}^{t} \mathrm{d}s \int_{-\infty}^{s} \mathrm{d}s' \, e^{-\nu k(t-s)} e^{-\nu k |s'|} \\ \cdot \int_0^\infty \mathrm{d}q \, q^{d+1} q^\alpha q^{2n} e^{-Tq^2/k} e^{-A_q(s-s')} \\ = J_1 + J_2 + J_3, \end{aligned} \tag{4.22}$$

where the three terms correspond to integrating over the regions with $s, s' < 0$, $s > 0, s' < 0$ and $s, s' > 0$ respectively. It is straightforward to see that $J_1 \sim e^{-\nu k t}$. For J_2 substitute $s' \to -s'$. Then, for model A,

$$J_2 = \frac{\nu D T}{k} \int_0^t \mathrm{d}s \int_0^\infty \mathrm{d}s' \, e^{-\nu k t} e^{\nu k(s-s')} e^{-Dr(s+s')} \int \frac{\mathrm{d}^d q}{(2\pi)^d} q_i^2 q^{2n} e^{-Tq^2/k} e^{-Dq^2(s+s')} \tag{4.23}$$

The dominant contribution comes from $s = 0$, $s' = 0$ if $Dr > \nu k$, and from $s = t$, $s' = 0$ for $Dr < \nu k$. For $Dr > \nu k$ then the temporal integral is asymptotic to the integrand evaluated in $s = 0$, $s' = 0$, and one gets $J_2 \sim e^{-\nu k t}$. For $Dr < \nu k$, instead, one gets e^{-Drt}. In the case $r = 0$

$$J_2 \sim \int_0^\infty \mathrm{d}q \, q^{d+2n+1} e^{-Tq^2/k} e^{-Dq^2 t} \tag{4.24}$$

$$= t^{-(d/2+n+1)} \int_0^\infty \mathrm{d}q \, q^{d+2n+1} e^{-Tq^2/kt} e^{-Dq^2} \tag{4.25}$$

$$\sim t^{-(d/2+n+1)}, \tag{4.26}$$

where we rescaled $q \to t^{-1/2}q$. For model B the procedure is very similar. For $r > 0$, $J_2 \sim t^{-(d/2+n+2)}$ while for $r = 0$ one has $J_2 \sim t^{-(d/4+n/2+1)}$, which can obtained by rescaling $q \to t^{-1/4}$. For J_3 make the substitution $u = s - s'$, $v = s - s'$ and integrate over v. For model A one gets

$$J_3 \sim \int_0^t \mathrm{d}u\,(t-u)e^{-\nu kt}e^{(\nu k - Dr)u}\int_0^\infty \mathrm{d}q\,q^{d+2n+1}e^{-Tq^2/k}e^{-Dq^2u}. \tag{4.27}$$

With similar reasoning as before one gets that $J_3 \sim e^{-\nu kt}$ for $Dr > \nu k$, $J_3 \sim e^{-Drt}$ for $Dr < \nu k$ and $J_3 \sim t^{-(d/2+n+1)}$ for $r = 0$. For model B instead one has, with the same substitution,

$$J_3 \sim \int_0^t \mathrm{d}u\,(t-u)e^{-\nu kt}e^{\nu ku}\int_0^\infty \mathrm{d}q\,q^{d+2n+3}e^{-Tq^2/k}e^{-Dq^2(r+q^2)u}, \tag{4.28}$$

which gives $J_3 \sim t^{-(d/2+n+2)}$ for $r > 0$ and $J_3 \sim t^{-(d/4+n/2+1)}$ for $r = 0$.

Therefore, the expansion for model A is

$$\langle y^{(2)}(t)\cdot z^{(0)}(0)\rangle \sim \begin{cases} e^{-\nu kt} & \text{for} \quad Dr > \nu k, \\ e^{-Drt} & \text{for} \quad Dr < \nu k, \\ t^{-(d/2+n+1)} & \text{for} \quad r = 0, \end{cases} \tag{4.29}$$

while for model B

$$\langle y^{(2)}(t)\cdot z^{(0)}(0)\rangle \sim \begin{cases} t^{-(d/2+n+2)} & \text{for} \quad r > 0, \\ t^{-(d/4+n/2+1)} & \text{for} \quad r = 0. \end{cases} \tag{4.30}$$

In the case of identical particles, by symmetry, one can trivially sees that

$$\langle y^{(0)}(t)\cdot z^{(2)}(t')\rangle = \langle y^{(2)}(t')\cdot z^{(0)}(t)\rangle. \tag{4.31}$$

It is straightforward to see that $\langle y^{(0)}(t)\cdot z^{(2)}(0)\rangle \sim e^{-\nu k|t|}$ as $t \to -\infty$ for both model A and model B and for any choice of the parameters, so that this term is always subleading compared to the other that enter the cross-correlation.

In conclusion, two identical colloids placed in the same fluid undergoing a phase transition, become correlated by the presence of the medium. Their cross correlation is non-zero and is at long times, in the case of a model A dynamics for the order parameter,

$$\langle y(t)\cdot z(0)\rangle \sim \lambda^2 \begin{cases} e^{-\nu kt} & \text{for} \quad Dr > \nu k, \\ e^{-Drt} & \text{for} \quad Dr < \nu k, \\ t^{-d/2} & \text{for} \quad r = 0, \end{cases} \tag{4.32}$$

while for model B dynamics

$$\langle y(t) \cdot z(0) \rangle \sim \lambda^2 \begin{cases} t^{-(d/2+1)} & \text{for} \quad r > 0, \\ t^{-d/4} & \text{for} \quad r = 0. \end{cases} \tag{4.33}$$

We notice that these exponents are exactly those of a the autocorrelation of a single particle, and the qualitative consideration presented in Section 3.4 apply here as well. At present, we are not able to give a solid theoretical argument for the similarity between these exponents.

Chapter 5

Conclusions and perspectives

In this thesis we studied how the dynamics of a Brownian particle is modified when the fluid in which it is suspended undergoes a continuous phase transition. Our model can also describe similar systems; for example, a mesoscopic impurity diffusing in a ferromagnet near a Ising-like critical point.

We showed that by assuming a linear coupling between the particle and the order parameter, modeled as a scalar field in the Gaussian approximation, static equilibrium properties are unaffected by the transition.

Then, working perturbatively in the strength of the coupling we showed instead that dynamical properties are influenced by it:

- the stationary state autocorrelation of a harmonically confined particle decays algebraically at long times as $t^{-d/2}$ at the critical point for a purely relaxational (model A) dynamics of the order parameter; if the dynamics of the order parameter satisfies a global conservation law (model B) then this decay becomes $t^{-d/4}$ at criticality and persist also away from it, but with a different exponent $t^{-(d/2+1)}$;
- the relaxation at equilibrium of a harmonically trapped particle decays $t^{-(d/2+1)}$ for a critical model A dynamics, and as $t^{-(d/4+1)}$ and $t^{-(d/2+2)}$ for a critical and non-critical model B dynamics respectively;
- the cross-correlation in the stationary state between two trapped particles decays algebraically, with the same exponents as the single-particle autocorrelation;
- all these exponents were shown to be universal, in the sense of being independent on the microscopic details of the interaction between the particle and the field;
- these properties were shown to remain valid also in a non-perturbative numerical simulation of the system.

Still, many issues are still to be addressed. Within the framework of the thesis, it remains unclear whether these properties persist also in a non-perturbative regime. Moreover there are evident relations among the derived exponents, and between these and the critical ones characterizing the transition, that are still to be explained non-heuristically.

Among possible extension of the setting studied in this work we note the possibility of considering a non-Gaussian order parameter. As discussed in the introduction, non-linearities are not irrelevant in a renormalization group sense and neglecting them is a rather crude approximation [53].

Further attention should also be devoted to the type of coupling between the particle and the order parameter. For example, it is known that a quadratic coupling is needed to model a stiff protein diffusing on a membrane [64].

Moreover, we assumed the order parameter to be at equilibrium at all times, but the possibility of considering non-equilibrium scenarios such as order-parameter quenches would be interesting to examine, as they appear to be a rewarding topic [57].

Appendix A

Subleading terms in the autocorrelation

We consider here the asymptotic expansion of Eq. 3.9. For simplicity we consider two terms $\langle y_i^{(2)}(t) y_i^{(0)}(0)\rangle = I_1 + I_2$ separately.

First term

The first of the two is

$$I_1 = \frac{\nu C_d}{k} \int_{-\infty}^{t} \mathrm{d}s \, e^{-\nu k(t-s)} \int_{-\infty}^{s} \mathrm{d}s' \, (e^{-\nu k|s'|} - e^{-\nu k|s|}) \\ \cdot \int_0^\infty \mathrm{d}q \, q^{d+1} |V_q|^2 A_q Q_q^{\mathrm{eq}}(s-s') G_q(s-s') \tag{A.1}$$

Given the presence of the absolute values in the integrand, for model A it's easier to study separately three terms $I_1 = I_{1a} + I_{1b} + I_{1c}$ corresponding to integrating to three different regions in the time integrals: $s, s' \in (-\infty, 0]$ for I_{1a}, $s \in [0,t]$, $s' \in (-\infty, 0]$ for I_{1b} and $s \in [0,t]$, $s' \in [0,s]$ for I_{1c}.

It is trivial to see that $I_{1a} \sim e^{-\nu k t}$ since there is no dependence on t in the integration region.

I_{1b} is explicitly

$$I_{1b} \propto \int_0^t \mathrm{d}s \int_0^\infty \mathrm{d}s' \, e^{-\nu k(t-s)} (e^{\nu k s'} - e^{-\nu k s}) e^{-Dr(s-s')} \\ \cdot \int_0^\infty \mathrm{d}q \, q^{d+1} |V_q|^2 (r+q^2) Q_q^{\mathrm{eq}}(s-s') e^{-Dq^2(s-s')} \tag{A.2}$$

The part of the integrand that sets dominating values at large times is $(e^{\nu k(s+s')} - 1)e^{-Dr(s-s')}$. For $Dr > \nu k$ therefore to get the asympotic behaviour one can set evaluate the integrand in $s' = 0$, then take its derivative in $s = 0$. Doing so one gets $I_{1b} \sim e^{-\nu k t}$.

For $Dr < \nu k$ the integral is simply asymptotic to the integrand evaluated in $s = t$, $s' = 0$

$$I_{1b} \sim e^{-Drt} \int_0^\infty \mathrm{d}q\, q^{d+1+2n}(r+q^2)e^{-Tq^2/k(1-e^{-\nu kt})}e^{-Dq^2 t} \tag{A.3}$$

By rescaling $q \to t^{1/2}q$ one obtains that $I_{1b} \sim e^{-Drt}$ for finite r and $t^{-(d/2+n+1)}$ for $r = 0$. Therefore, for both critical and non-critical model A I_{1b} term has the same asymptotic as $\langle y^{(1)}(t)y^{(1)}(0)\rangle$.

The integrand of I_{1c} depends only on the difference $s-s'$. Substituing $u = s-s'$, $v = s+s'$ it is immediate to see that the part of the integrand that gives the main contribution is $(e^{\nu ku} - 1)e^{-Dru}$. The integral is then asymptotic to the derivative of the integral evaluated in $u = 0$ if $Dr > \nu k$, which gives $e^{-\nu kt}$ for this range of the parameters, and to the integral evaluated in $u = t$ if $Dr < \nu k$, which gives $I_{1c} \sim e^{-Drt}$ for $0 < Dr < \nu k$ and $I_{1c} \sim t^{-(d/2+2+n)}$. Therefore for model A I_{1c} is the same order of the other terms in the non-critical case and subleading in the critical case.

The asymptotic of I_1 is easier to deduce for model B. In this case

$$\begin{aligned} I_1 \propto \int_{-\infty}^t \mathrm{d}s \int_{-\infty}^s \mathrm{d}s'\, e^{-\nu k(t-s)}(e^{-\nu k|s'|} - e^{-\nu k|s|}) \\ \cdot \int_0^\infty \mathrm{d}q\, q^{d+3+2n}(r+q^2)Q_q^{\mathrm{eq}}(s-s')G_q(s-s') \end{aligned} \tag{A.4}$$

The dominating values in the time integrals are set by $e^{\nu ks}(e^{-\nu k|s'|} - e^{-\nu k|s|})$, which is rapidly decaying away from $s = t$, $s' = 0$. Asymptotically then we can simply evaluate the integrand in this point

$$I_1 \sim (1 - e^{-\nu kt}) \int_0^\infty \mathrm{d}q\, q^{d+3+2n}(r+q^2)Q_q^{\mathrm{eq}}(t)G_q(t) \tag{A.5}$$

By usual arguments $I_1 \sim t^{-(d/2+2+n)}$ for $r > 0$ and $I_1 \sim t^{-(d/4+3/2+n/2)}$, which is in both cases subleading.

Second term

The second term is

$$\begin{aligned} I_2 = \frac{\nu^2 TC_d}{k} \int_{-\infty}^t \mathrm{d}s\, e^{-\nu k(t-s')} \int_{-\infty}^s \mathrm{d}s'\, (e^{-\nu k|s'|} - e^{-\nu k|s|}) \\ \cdot \int_0^\infty \mathrm{d}q\, q^{d+3}|V_q|^2 Q_q^{\mathrm{eq}}(s-s')G_q(s-s') \end{aligned} \tag{A.6}$$

Notice that first term is $e^{-\nu k(t-s')}$, which is different from the corresponding term in Eq. (A.1).

For model A we proceed by considering the three integrals $I_2 = I_{2a} + I_{2b} + I_{2c}$ over the same three regions as done with I_1. $I_{2a} \sim e^{-\nu kt}$ for the same reasons as I_{1a}. For I_{2b}

and I_{2c} the important part of the integrand is $e^{\nu k s'}(e^{-\nu k|s'|} - e^{-\nu k|s|})e^{-Dr(s-s')}$. This term implies that I_{2b} is asymptotic to the derivative with respect to s of the integrand evaluated in $s, s' = 0$, which gives $I_{2b} \sim e^{-\nu k t}$ for all values of the parameters.

For I_{2c} substituting $u = s - s'$, $v = s + s'$ and integrating over v gives an integrand depending only on u. The aforementioned term implies that for $r > 0$ the integral is proportional to the derivative w.r.t. to u evaluated in $u = 0$, which gives $I_{2c} \sim e^{-\nu k t}$. For $r = 0$ this term is approximately constant over the integration region and by substituing $q \to t^{1/}2q$ one gets $I_{2c} \sim e^{-\nu k t}$ also in this case.

For model B instead the asymptotic behaviour of I_2 can be found with a very similar procedure as done for I_1. The only difference from from before is that there is no dominant value of s and s' in the integrand set by the term $e^{\nu k s'}(e^{-\nu k|s'|} - e^{-\nu k|s|})$, but it's vanishingly small outside the region $s, s' \in [0, t]$, where is approximately constant. With this observation, it is straightforward to conclude that also for model B $B \sim e^{-\nu k t}$ for any value of the parameters.

Appendix B

Path-integral calculation of the diffusion constant

In this Appendix we use a path-integral method to compute the diffusion constant of a free (i.e., $k = 0$) particle coupled to the field, as done in Section 2.6 with another approach. The calculations are similar to those in Ref. [48].

Preliminaries

The equation of motion of the field Eq. (1.17) can be formally solved as

$$\phi(x,t) = \int_{-\infty}^{t} \mathrm{d}s\, e^{-A(t-s)} \left[\zeta(x,s) + \lambda D V(x - y(s))\right] \tag{B.1}$$

This expression can be then inserted in the equation of motion for the particle Eq.(1.18) to get a closed equation for the particle

$$\dot{y}(t) = \xi(t) + \int_{-\infty}^{t} \mathrm{d}s\, \gamma(y(t) - y(s), t - s) + \eta(y(t), t) \tag{B.2}$$

where we defined

$$\begin{aligned} \gamma(x,t) &\equiv \nu\lambda^2 D \nabla V e^{-At} V(x) \\ \eta(x,t) &\equiv -\nu\lambda \int_{-\infty}^{t} \mathrm{d}s\, \nabla V e^{-A(t-s)} \zeta(x,s) \end{aligned} \tag{B.3}$$

The γ term of Eq. (B.1) is a sort of memory-kernel and describes the effective interaction of the particle with itself at different times through the field. This makes the dynamics of the particle manifestly non-Markovian.

The noise $\eta(x,t)$ represent the effect of the noise $\zeta(x,t)$ that acts on the field and is mediated by the latter to the particle. Its presence is needed to ensure the correct equilibrium distribution. Being a linear combination of the gaussian noise $\zeta(x,t)$ at different times and points in space, it is itself gaussian [59]. Its mean vanishes

$$\langle \eta(x,t) \rangle = 0 \tag{B.4}$$

To compute its correlations start by noticing that for generic translationally invariant operators A, B and a function satisfying $\langle\zeta(x)\zeta(x')\rangle = C(x-x')$

$$\langle (A\zeta)(x)(B\zeta)(x')\rangle = \int \mathrm{d}^d y\,\mathrm{d}^d y'\, A(x-y)B(x'-y')\langle\zeta(y)\zeta(y')\rangle \tag{B.5}$$

$$= \int \mathrm{d}^d y\,\mathrm{d}^d y'\, A(x-y)B(x'-y')C(y-y') \tag{B.6}$$

$$= ABC(x-x') \tag{B.7}$$

and that for a generic operator A

$$\int \mathrm{d}t\, e^{At} = e^{At}A^{-1} \tag{B.8}$$

With these and some calculations

$$\begin{aligned}\langle\eta_i(x,t)\eta_j(x',t')\rangle &= \nu^2\lambda^2 \int_{-\infty}^{t} \mathrm{d}s \int_{-\infty}^{t'} \mathrm{d}s'\, \langle \nabla_i V e^{-A(t-s)}\zeta(x,s)\ \nabla'_j V e^{-A(t'-s')}\zeta(x',s')\rangle \\ &= H(x-x', t-t')\end{aligned} \tag{B.9}$$

where

$$H(x,t) = -\lambda^2\nu^2 DT\nabla_i\nabla_j V^2 e^{-A|t|}A^{-1}(x) \tag{B.10}$$

which is to be compared with Eq. (14) of reference [48].

Derivation of the action

We work in the response function formalism, also known as Janssen-De Domincis-Peliti or Martin-Siggia-Rose formalism [65–67]. Denoting by $P[\xi]$ and $P[\eta]$ the functional probability distributions of the gaussian noises and with $y^{(\xi,\eta)}$ the solution of Eq. (B.2) for fixed configurations of the noises, the partition function of the system is

$$\begin{aligned} Z &= \int [\mathrm{d}y][\mathrm{d}\xi][\mathrm{d}\eta]\, P[\xi]P[\eta] \prod_t \delta\left[y - y^{(\xi,\eta)}\right] \\ &= \int [\mathrm{d}y][\mathrm{d}\xi][\mathrm{d}\eta]\, P[\xi]P[\eta] \\ &\qquad \cdot \prod_t \delta\left[\dot{y}(t) - \xi(t) - \int_0^t \mathrm{d}s\,\gamma(y(t)-y(s), t-s) - \eta(y(t),t)\right]\end{aligned} \tag{B.11}$$

In principle, a Jacobian should be included when passing between the two expression, similarly as it happens with a finite-dimensional δ-function [67]. However, employing the Ito convention, the transformation matrix is triangular with constant diagonal terms. Therefore the Jacobian is constant and has been ignored [48, 67].

The standard way to proceed is to introduce an auxilary field $p(t)$ to express the δ-functional in a tractable way

$$Z = \int [\mathrm{d}y][\mathrm{d}p][\mathrm{d}\xi][\mathrm{d}\eta]\, P[\xi]P[\eta] \\ \cdot \exp\left[i \int \mathrm{d}t\, p(t) \left(\dot{y}(t) - \xi(t) - \int_0^t \mathrm{d}s\, \gamma(y(t)-y(s), t-s) - \eta(y(t),t)\right)\right] \tag{B.12}$$

Since the noises are gaussian, the integration over ξ and η can be carried out explicitly. Omitting differentials for brevity

$$\int [\mathrm{d}\xi] P[\xi] \exp\left[i \int p(t)\xi(t)\right] = \frac{1}{Z_\xi} \int [\mathrm{d}\xi] \exp\left[-\frac{1}{4\nu T} \int \xi(t)^2\right] \exp\left[i \int p(t)\xi(t)\right] \\ = \exp\left[-\nu T \int p(t)^2\right] \tag{B.13}$$

and

$$\int [\mathrm{d}\eta] P[\eta] \exp\left[-i \int p(t)\eta(y(t),t)\right] \\ = \frac{1}{Z_\eta} \int [\mathrm{d}\eta] \exp\left[-\frac{1}{2} \int \eta(x,t) H^{-1}(x-x', t-t')\eta(x',t')\right] \exp\left[-i \int p(t)\eta(y(t),t)\right] \\ = \exp\left[-\frac{1}{2} \int p(t) H(t-t', y(t)-y(t'))p(t')\right] \tag{B.14}$$

where $Z_\xi = \int [\mathrm{d}\xi]P[\xi]$ and similarly for Z_η

With these the partition function now reads

$$Z = \int [\mathrm{d}y][\mathrm{d}p] e^{-S[y,p]} \tag{B.15}$$

where the action

$$S[y,p] = S_0[y,p] + S_{\mathrm{int}}[y,p] \tag{B.16}$$

is the sum of two parts. The first describing free brownian motion

$$S_0[y,p] = -i \int \mathrm{d}t\, p(t)\dot{y}(t) + \nu T \int \mathrm{d}t\, p(t)^2 \tag{B.17}$$

and the second describing the memory effects

$$S_{\mathrm{int}}[y,p] = \int \mathrm{d}t\, \mathrm{d}s \left[ip(t)\gamma(y(t)-y(s), t-s) + \frac{1}{2} p(t) H(y(t)-y(s), t-s) p(s)\right] \theta(t-s) \tag{B.18}$$

Free averages

For what follows we will need the free propagators. By usual results for brownian motion

$$\langle y(t)y(s)\rangle_0 = 2\nu T \min(t,s) \tag{B.19}$$

To obtain the correlations of the auxiliary field, start by noticing that if we add a forcing term to the free equation $\dot{y} = \xi(t) + f(t)$, then the action becomes

$$S'[y,p,f] = S_0[y,p] - i\int \mathrm{d}t\, p(t)f(t) \tag{B.20}$$

and the linear response for a generic observable O

$$\frac{\delta\langle O\rangle_f}{\delta f(t)} = i\langle Op(t)\rangle_0 \tag{B.21}$$

and from this

$$\langle p(t)\rangle_0 = -i\frac{\delta\langle 1\rangle_f}{\delta f(t)} = 0 \tag{B.22}$$

$$\langle p(t)p(s)\rangle_0 = -i\frac{\delta\langle p(t)\rangle_f}{\delta f(t)} = 0 \tag{B.23}$$

Alternatively to obtain p correlations notice

$$\frac{\delta S_0[y,p]}{\delta y(s)} = i\dot{p}(s) \tag{B.24}$$

so that

$$0 = \frac{1}{Z_0}\int [\mathrm{d}y][\mathrm{d}p]\frac{\delta}{\delta y(s)}\left(p(t)e^{-S_0[y,p]}\right) = -\frac{i}{Z_0}\int [\mathrm{d}y][\mathrm{d}p]p(t)\dot{p}(s)e^{-S_0[y,p]} \tag{B.25}$$

which implies that $\langle p(t)\dot{p}(s)\rangle = 0$ but, since the action is local, in turn implies

$$\langle p(t)p(s)\rangle = 0 \tag{B.26}$$

Cross-correlations can be obtained by noticing that

$$\begin{aligned} 0 &= \frac{1}{Z_0}\int [\mathrm{d}y][\mathrm{d}p]\frac{\delta}{\delta p(s)}\left(p(t)e^{-S_0[y,p]}\right) \\ &= \frac{1}{Z_0}\int [\mathrm{d}y][\mathrm{d}p]\Big(\delta(t-s) + ip(t)\dot{y}(s) - \nu T p(t)p(s)\Big)e^{-S_0[y,p]} \end{aligned} \tag{B.27}$$

which implies that $\langle p(t)\dot{y}(s)\rangle_0 = i\delta(t-s)$ or, integrating in s,

$$\langle y(t)p(s)\rangle_0 = i\alpha(t,s) \equiv i\begin{cases} 1 & 0 \leqslant s < t \\ 0 & \text{otherwise} \end{cases} \tag{B.28}$$

In what follows consider $t \geqslant s \geqslant 0$ and $T \geqslant 0$. We will need

$$\langle p(t)e^{ik(y(t)-y(s))}\rangle_0 = \sum_n c_n \langle p(t)(y(t)-y(s))^n\rangle_0 \propto \langle p(t)(y(t)-y(s))\rangle_0 = 0 \tag{B.29}$$

where c_n are unimportant coefficient. In a very similar manner

$$\langle p(t)p(s)e^{ik(y(t)-y(s))}\rangle_0 = 0 \tag{B.30}$$

Finally we will need

$$\begin{aligned}
\langle y(T)^2 p(t) e^{ik(y(t)-y(s))}\rangle_0 &= \sum_n \frac{(ik)^{2n+1}}{(2n+1)!}\langle y(T)^2 p(t)(y(t)-y(s))^{2n+1}\rangle_0 \\
&= \sum_n \frac{(ik)^{2n+1}}{(2n+1)!} 2(2n+1)\langle y(T)p(t)\rangle_0 \\
&\qquad \cdot \langle y(T)(y(t)-y(s))\rangle_0 \frac{(2n)!}{2^n n!}\langle (y(t)-y(s))^2\rangle_0^n \\
&= -\alpha(T,t) 4\nu T k (t-s) e^{-\nu T k^2 (t-s)}
\end{aligned} \tag{B.31}$$

and

$$\begin{aligned}
&\langle y(T)^2 p(t)p(s)e^{ik(y(t)-y(s))}\rangle_0 \\
&\quad = \sum_n \frac{(ik)^{2n}}{(2n)!}\langle y(T)^2 p(t)p(s)(y(t)-y(s))^{2n}\rangle_0 \\
&\quad = \sum_n \frac{(ik)^{2n}}{(2n)!} 2\langle y(T)p(t)\rangle_0 \langle y(T)p(s)(y(t)-y(s))^{2n}\rangle_0 \\
&\quad = 2i\alpha(T,t)\sum_n \frac{(ik)^{2n}}{(2n)!}\Bigg[\langle y(T)p(s)\rangle_0 \frac{(2n)!}{2^n n!}\langle (y(t)-y(s))^2\rangle_0^n \\
&\qquad\qquad + 2n(2n-1)\langle y(T)(y(t)-y(s))\rangle_0 \langle p(s)(y(t)-y(s))\rangle_0 \\
&\qquad\qquad\qquad \cdot \frac{(2n-2)!}{2^{n-1}(n-1)!}\langle (y(t)-y(s))^2\rangle_0^{n-1}\Bigg] \\
&\quad = 2i\alpha(T,t)\left[i\alpha(T,s) - 2\nu T k^2 \left(\min(T,t)-\min(T,s)\right) i\alpha(t,s)\right] e^{-\nu T k^2 (t-s)} \\
&\quad = \alpha(T,t)\left(4\nu T k^2 (t-s) - 2\right) e^{-\nu T k^2 (t-s)}
\end{aligned} \tag{B.32}$$

Effective diffusion constant

It is now straightforward to compute the effective diffusion constant

$$\langle y(T)^2\rangle = \frac{\langle y(T)^2 e^{-S_{\text{int}}}\rangle_0}{\langle e^{-S_{\text{int}}}\rangle_0} \approx \frac{\langle y(T)^2(1-S_{\text{int}})\rangle_0}{\langle 1-S_{\text{int}}\rangle_0} \tag{B.33}$$

Notice that

$$\gamma(x,t) = \int \frac{\mathrm{d}q}{2\pi} e^{iqx} \gamma(q,t) \qquad \gamma(q,t) = \lambda^2 \nu D i q V_q^2 e^{-A_q t} \tag{B.34}$$

$$H(x,t) = \int \frac{\mathrm{d}q}{2\pi} e^{iqx} H(q,t) \qquad H(q,t) = \lambda^2 \nu^2 D T \frac{q^2 V_q^2 e^{-A_q|t|}}{A_q} \tag{B.35}$$

Given the previous results it's immediate to conclude that $\langle S_{\mathrm{int}} \rangle_0 = 0$. The first non zero perturbative correction to the displacement is then given by

$$\begin{aligned} \langle y(T)^2 S_{\mathrm{int}} \rangle = \int \mathrm{d}t\, \mathrm{d}s \int \frac{\mathrm{d}q}{2\pi} \Big[& i\gamma(q, t-s) \langle y(T)^2 p(t) e^{iq(y(t)-y(s))} \rangle_0 \\ & + \frac{1}{2} H(q, t-s) \langle y(T)^2 p(t) p(s) e^{iq(y(t)-y(s))} \rangle_0 \Big] \end{aligned} \tag{B.36}$$

and putting all together one finally gets

$$D - D_0 = \lim_{T\to\infty} \frac{\langle y(T)^2 \rangle}{T} - D_0 = -\lambda^2 \int \frac{\mathrm{d}q}{2\pi} \frac{2\nu^2 T q^2 V_q^2}{A_q(\nu T q^2 + D V_q A_q)} + o(\lambda^2) \tag{B.37}$$

which is the same result as Eq. (2.53).

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