

"pure lines" which differ among themselves but are *per se* constant in size, but, due to the peculiar way in which rats receive their infections, a given infection consists of only a very few of these lines. In fact, a large number of "wild" infections are probably actual "pure lines."

¹ Throughout this work the term "pure line infection" is used to designate an infection, the trypanosomes of which have all arisen from a single organism. A given "pure line" may either have been started from a single trypanosome, or it may have been subinoculated from such an infection. The term "wild infection" designates an infection as found in nature.

² These PROCEEDINGS, 7, 1921 (138-143).

³ The mean sizes in this report are all in microns.

⁴ Minchin, E. A., and Thomson, J. D., *Quart. J. Microsc. Sci.*, 60, N. S., 1915 (463-692).

⁵ Gondor, R., *Centralbl. Bakt., etc.*, I abt., Originale, 612, 1911 (102-113).

⁶ Robertson, M., *Proc. Roy. Soc.*, (B) 85, 1912 (241-248).

⁷ Jennings, H. S., *J. Exper. Zool.*, 11, 1911 (1-134); *Ibid.*, 14, 1913 (270-391).

⁸ Erdmann, R., *Archiv. Entwicklungsmech. Organ.*, 46, 1920 (85-148).

NOTE ON MOVING EQUILIBRA*

BY ALFRED J. LOTKA

BROOKLYN, N. Y.

Communicated by R. Pearl, March 12, 1921

A number of previous publications¹ have been devoted to the study, from various angles, of a material system evolving in accordance with a system of differential equations

$$dX_i / dt = F_i(X_1, X_2, \dots; A_1, A_2, \dots; P; Q) \quad (1)$$

where the X 's denote the masses of the several components of which the system is built up; the A 's are parameters introduced by any equations of constraint to which the X 's may be subject;² the parameters P include geometrical constraints (volume, area, topography) and also other quantities serving to define the state of the system (temperature, etc.). The Q 's define the character of the several components or species.

The discussion has hitherto been restricted to the case that the parameters A , P , Q remained constant during the transformations taken in view. A complete discussion of the evolution of systems of the kind referred to must include also the consideration of changes in these parameters.

Such changes may be grouped under three heads:

1. Changes of a perfectly general character. A study of these would resolve itself into a discussion, on a general basis, of a system of differential equations of the form

$$dX_i / dt = F_i(X_1, X_2, \dots, t) \quad (2)$$



It is not intended to enter here into this perfectly general case. It will suffice to point to the purely mathematical literature on the subject.³

2. Changes in the parameters A , P , Q proceeding very slowly as compared with the speed of readjustment of the X 's. In this case the system, after passing through a "transient" state, ultimately settles down to a "moving equilibrium." The study of this case therefore comprises two phases, which may with advantage be taken up separately. The phase of the moving equilibrium is of particular interest, since such moving equilibria play an important rôle in the evolution of physical systems, as pointed out years ago by Herbert Spencer.⁴

3. We may study the effect of a change in a parameter A , P or Q upon the equilibrium of the system alone, irrespective of the process by which that equilibrium is reached. It is in this case immaterial whether the change is slow or rapid. Into this division of the subject fall such relations as the principle of Le Chatelier, the thermodynamic laws of equilibrium and the "reciprocal relations" of statistical mechanics.

The application of some of these principles to biological and social systems has been essayed, but it cannot be said that the rigor of the attempts thus made is satisfactory. It would therefore be desirable to go over the ground and consolidate it. An effort in this direction is taken in view as part of the plan into which the present contribution is fitted.⁵

Of the general field outlined above, the portion to which we shall now give our attention is that of moving equilibria.

Our fundamental system of equations we shall, for our present purpose, write in the form (2). Furthermore, merely in order to simplify our notation, we will restrict the number of dependent variables to two, which, to avoid subscripts, we will denote by X , Y . We have, then

$$dX/dt = F_1(X, Y, t), \quad dY/dt = F_2(X, Y, t). \quad (3)$$

We adopt a method of successive approximations. Since the system is near equilibrium, we write for our first approximation

$$0 = F_1(X, Y, t), \quad 0 = F_2(X, Y, t). \quad (4)$$

Solving for X and Y we then have

$$X_1 = \varphi_1(t), \quad X_1 = \psi_1(t). \quad (5)$$

whence by differentiation

$$dX_1/dt = X_1' = \varphi_1'(t), \quad dY_1/dt = Y_1' = \psi_1'(t). \quad (6)$$

Proceeding to a second approximation, we substitute (6) in (3).

$$\varphi_1'(t) = F_1, \quad \psi_1'(t) = F_2. \quad (7)$$

Solving again for X , Y ,

$$X_2 = \varphi_2(t), \quad Y_2 = \psi_2(t). \quad (8)$$

We may again differentiate,

$$X_2' = \varphi_2'(t), \quad Y_2' = \psi_2'(t). \quad (9)$$

and, substituting as before, we obtain a third approximation. We continue this process as far as may be desired, and finally obtain for the n th

approximation

$$X_n = \varphi_n(t), \quad Y_n = \psi_n(t). \quad (10)$$

If the functions φ_n , ψ_n , and φ'_n , ψ'_n , obtained by these successive approximations tend toward a limit as n is increased indefinitely, then it can easily be shown that (10) is a solution of (3).

It seems to be somewhat difficult to give in general terms the conditions for this convergence of successive approximations toward a limit. However, the following example will show how in individual cases the question regarding this convergence can sometimes be readily answered.

For our example we take the case of radioactive equilibrium. We have a chain of transformations



We will denote the masses of A , B , C , D at time t , respectively, by U , X , Y , Z , and their values at time $t=0$ by the same letters with the subscript zero.

We may if we choose (this is a purely arbitrary matter) pick out the substances B and C for our evolving system, and look upon A and D as external factors influencing the system. The system composed of B and C is then subject to an equation of constraint

$$X + Y = X_0 + Y_0 + U_0 - U + Z_0 - Z \quad (12)$$

in which the quantities appearing in the right hand member are parameters of the class denoted above in equation (1) by A , that is to say, parameters introduced by the equation of constraint. Four of them are constants, the other two are functions of t . Of these last two one, namely U , will appear in the equations representing the course of evolution of the system composed of B and C . These equations, according to the well-known laws of radioactive transformations are

$$dX/dt = aU - bX, \quad dY/dt = bX - cY. \quad (13)$$

where a , b , c are constants. U , on the other hand, is a function of the time, namely

$$U = U_0 e^{-at} \quad (14)$$

For our first approximation we put, then,

$$dX/dt = 0 = aU_0 e^{-at} - bX_1 \quad dY/dt = 0 = bX_1 - cY_1 \quad (15)$$

whence

$$X_1 = a/b \cdot U_0 e^{-at}, \quad Y_1 = a/c \cdot U_0 e^{-at}, \quad (16)$$

and therefore

$$X'_1 = -a^2/b \cdot U_0 e^{-at}, \quad Y'_1 = -a^2/c \cdot U_0 e^{-at}. \quad (17)$$

The second approximation now gives

$$dX/dt = X'_1 = -a^2/b \cdot U_0 e^{-at} = aU_0 e^{-at} - bX_2 \quad (18)$$

$$dY/dt = Y'_1 = -a^2/c \cdot U_0 e^{-at} = bX_2 - cY_2 \quad (19)$$

Whence

$$X_2 = a/b \cdot U_0 e^{-at} (1 + a/b) \quad (20)$$

$$Y_2 = a/c \cdot U_0 e^{-at} (1 + a/b + a/c) \quad (21)$$

Continuing this process we find, ultimately,

$$X_n = a/b U_0 e^{-at} (1 + a/b + a^2/b^2 + \dots) \quad (22)$$

$$Y_n = a/c U_0 e^{-at} (1 + a/b + a/c + a^2/b^2 + a^2/bc + a^2/c^2 + \dots) \quad (23)$$

In this example the condition for the convergence of the successive approximations is immediately apparent. We must have

$$a/b < 1, \quad a/c < 1; \quad (24)$$

that is to say, the parent substance *A* must have a smaller decay constant than any of the succeeding members of the series. For obvious reasons this condition is always satisfied in natural radioactive mixtures.⁶

The series (22), (23) bring out the relation between the uncorrected equilibrium, as commonly computed on the assumption of constancy of mass of the parent substance, and the true equilibrium. The first-mentioned (for which Rutherford has suggested the term "secular equilibrium") is represented by the first term of the series. As Rutherford points out, the error of the first approximation, i.e. the difference between the secular and the true equilibrium, amounts, in some cases, to nearly 1% though in others the error is quite negligible.

The series are easily summed, and then lead to the well-known expressions obtained by other methods (for the equations of radioactive change are readily integrable in finite terms, while the method here developed is applicable also in more refractory cases).

The case of radioactive equilibrium was here selected as an illustration, primarily because the functions involved are known and of simple form. But the same example will serve very aptly to illustrate also some other points.

In the first place we observe that moving equilibria might be divided into three classes, according as their progress is determined by a change in the *P*'s, the *Q*'s or the *A*'s. As has been shown, the radioactive equilibrium is of the type in which the pace is set by a parameter of the class *A*, namely *the mass of one of the links in the chain*, which thus acts as a brake, or a *limiting factor* checking the series of transformations. Such limiting factors play an important rôle also in the highly complex network of interlocking cycles upon which the continuance of abundant life upon the earth depends. For life processes are energy transformation processes carried out by the agency of material energy transformers. Such transformers, if they are to work continuously and indefinitely must perforce work in closed transformation chains or cycles (such as the cycle $\text{CO}_2 \rightarrow \text{Plant} \rightarrow \text{Animal} \rightarrow \text{CO}_2$). The moving equilibria engendered in such systems of cycles by a slow change in a limiting factor, in a parameter of class *A*, invite further study. The influence of man upon the world's events seems to have been largely to accelerate the circulation of matter and energy through such cycles, either by "enlarging the wheel", i.e., increasing the mass taking part in certain cycles, or else by causing it to

"spin faster," i.e., increasing the velocity of the circulation, decreasing the time required for a given mass to complete the cycle. In either case he has increased the energy turn-over per unit of time. Whether, in this he has been unconsciously fulfilling one of those laws of nature according to which certain quantities tend toward a maximum, is a question well deserving of our attention.

* Papers from the Department of Biometry and Vital Statistics, School of Hygiene and Public Health, Johns Hopkins University, No. 44.

¹Lotka, A. J., *Physic. Rev.*, **24**, 1912 (235-238); *J. Washington Acad. Sci.*, **2**, 1912 (2, 49, 66); *Science Progress*, **55**, 1920 (406-417); *Proc. Am. Acad. Arts Sci.*, **55**, 1920 (237-153); these PROCEEDINGS, *Sci.* **6**, 1910 (410-415).

²*Proc. Amer. Acad.*, loc cit., p. 142.

³See for example Picard, *Traité d'Analyse*; H. Bateman, *Differential Equations*, 1918, p. 245.

⁴Spencer, *First Principles*, Chapter XXII; Winiarskie, "Essai sur la Mécanique Sociale," *Revue Philosophique*, **44**, 1900 (113).

⁵At the time of reading proof this project is partially realized. A discussion of the applicability of the Le Chatelier principle to systems of the general character here considered will appear in a forthcoming issue of the *Proceedings of the American Academy of Arts and Sciences*

⁶Lotka, A. J., *London, Phil. Mag.*, Aug., 1911, p. 353.

A FORMULA FOR THE VISCOSITY OF LIQUIDS¹

BY H. B. PHILLIPS

DEPARTMENT OF MATHEMATICS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Communicated by A. G. Webster, March 22, 1921

1. In this paper I obtain for the viscosity of a liquid the formula

$$\eta = \frac{n N h}{2M(v-\delta)}, \quad (1)$$

where N is the number of molecules in a mol, h is Planck's constant, M is the molecular weight of the liquid in the gas phase, v its volume per gram, and n an integer. The quantity δ is the co-volume as used in the equation of state of Keyes¹

$$p = \frac{RT}{v-\delta} - \frac{A}{(v-l)^2}, \quad (2)$$

In all the cases to which I have applied the formula, $n=6$ and so (1) takes the form

$$\eta(v-\delta) = 3Nh/M \quad (3)$$

It is to be noted that $3N/M$ is the number of translational degrees of freedom of the molecules in the volume v of the liquid.

2. To prove equation (1), let x, y, z be rectangular coördinates and suppose the liquid to flow parallel to the x -axis in such a way that, u_0 being